

Geometry Method for the Rotating Navier-Stokes Equations With Complex Boundary and the Bi-Parallel Algorithm *

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Abstract: In this paper, a new algorithm based on differential geometry viewpoint to solve the 3D rotating Navier-Stokes equations with complex Boundary is proposed, which is called Bi-parallel algorithm. For example, it can be applied to passage flow between two blades in impeller and circulation flow through aircrafts with complex geometric shape of boundary. Assume that a domain in R^3 can be decomposed into a series sub-domain, which is called "flow layer", by a series smooth surface $\mathfrak{S}_k, k = 1, \dots, M$. Applying differential geometry method, the 3D Navier-Stokes operator can be split into two kind of operator: the "Membrane Operator" on the tangent space at the surface \mathfrak{S}_k and the "Bending Operator" along the transverse direction. The Bending Operators are approximated by the finite different quotients and restricted the 3D Navier-Stokes equations on the interface surface \mathfrak{S}_k , a Bi-Parallel algorithm can be constructed along two directors: "Bending" direction and "Membrane" directors. The advantages of the method are that: (1) it can improve the accuracy of approximate solution caused of irregular mesh nearly the complex boundary; (2) it can overcome the numerical effects of boundary layer, which is a good boundary layer numerical method; (3) it is sufficiency to solve a two dimensional sub-problem without solving 3D sub-problem.

Key Words Impeller; Flow Passage; Geometry Method; Differential Geometry; Rotating Navier-Stokes equations; Bi-Parallel Algorithm.

Subject Classification(AMS): 65N30, 76U05, 76M05

1 Introduction

As well known, numerical simulation for 3D viscous flow in turbo-machinery and circulation flow through aircraft meet three kinds of difficulty cause of nonlinearity, three dimensional grid and boundary layer effects with complex boundary. In order to overcome last two difficulties, a new dimensional slitting method based on differential geometry method is proposed.

As well known that classical domain decomposition method is that the 3D domain is made in the sum of severest overlap or non overlap subdomain, an approximated solutions can be established then by solving 3D subproblem in each subdomain, see e.g., [2,4,19,24]. The method proposals by authors here, called "bi-parallel algorithm", for 3D Navier-Stokes equations, is that the 3D domain Ω occupied by the fluids in \mathfrak{R}^3 is decomposed into the sum of serverly sub domains (called layer) Ω_{i-1}^i by serverly 2D surfaces(2D manifold) $\mathfrak{S}_i, i = 1, 2, \dots, m$. 3D Navier-Stokes operators in the layer $\Omega_{i-1}^i \cup \Omega_i^{i+1}$ under a new coordinate based on the surface \mathfrak{S}_i can be represented as the sum of "membrane" operators on tangent space at \mathfrak{S}_i and normal (bending) operator to \mathfrak{S} . By Euler central difference quotient instead of bending operator and restricting 3D Navier-Stokes equations

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on the \mathfrak{S}_i , a three components and two dimensional Navier-Stokes equations (is called 3C-2D NSE) on \mathfrak{S}_i are obtained. After successively iterations, an approximate solution of 3D NSE can be established. It is obvious that the method is different from the classical domain decomposition method because we only solve a two dimensional problem in each sub domain without solving 3D subproblems. In addition, other advantages of this method are the followings:

(i) For the complex boundary geometry, for examples, in turbo-machinery flow with complex shape of blades of impeller, in geophysical flow with real surface of the earth and in the circulation flow passing through complex aircraft etc., 80 percent of the freedoms of 3D mesh should be concentrated on a thin domain of the boundary layer, even using different methods or finite element method, there exists a lot of difficulty and even influence of the accuracy ; in our method the distance between two surface can be very small as you wanted, owing to parametrization of surface, the domain for 2D-3C subproblem is a bounded domain in \mathbb{R}^2 , and the mesh is 2D-mesh;

(ii) Since interface surfaces are chosen such that most of flux flows along the interface while small flux of the fluid run cross the interface only, iterative method is very well suitable for the physical properties of flow;

(iii) This is a better method for treatment of boundary layer phenomenons, which can refer to [19] and references therein. Indeed this is a good boundary layer model and associated algorithm.

This paper is organized as follows: in section 2 we give the preliminary for the geometry of the blade's surface and construct a new coordinate system; in section 3, we derive the rotating Navier-Stokes equations in the new coordinate system; in section 4, we study The equations for the average velocity along the rotating direction; in section 5, provided the equations of the Gâteaux Derivatives of the solutions of Navier-stokes equations with respect to shape of boundary; in section 6 we provide 2D-3C Navier-Stokes equations on the Surface \mathfrak{S}_ξ ; in section 7, we provide a corrected equation for the pressure in order to improvement of the accuracy of the pressure's computation; in section 8, we present the Bi-parallel Algorithm; in section 9, we prove the existence of the solutions for 2D-3C Navier-Stokes equations and the solution of correcting equation for the pressure, and discuss the dependence of the corrected pressure upon the velocity; and finally, an appendix is supplied.

2 Preliminary—The Geometry of the Blade's Surface

we use Einstein summation convention throughout this paper, in order to distinguish, Greek indices and exponents belong to the set $\{1, 2\}$, Latin indices and exponents belong to the set $\{1, 2, 3\}$, and the repeated index implies that we are summing over all of its possible values. The dot product and the cross product of two vectors $\mathbf{a}, \mathbf{b} \in R^3$ are denoted by $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \times \mathbf{b}$, respectively, and the Euclidean norm $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$.

Let D be an open bounded connect subset of R^2 , whose boundary γ is Lipschitz-continuous. Suppose D is locally on one side of γ . In the following of this paper, we also suppose that the blade of impeller is so thin that it can be described as an smoothing injective mapping $\mathfrak{S} = \mathfrak{R}(D)$ in mathematics. In this case, D is the projection area of blade \mathfrak{S} on the meridian plane of impeller. Let $x = (x^\alpha)$ is an arbitrary point in the set \overline{D} , and $\partial_\alpha = \partial/\partial x^\alpha$. If the mapping $\mathfrak{R}(x)$ is smoothing enough, for example, $\mathfrak{R}(x) \in C^3(\overline{D}; R^3)$, then there exist two vectors $\mathbf{e}_\alpha(x) = \partial_\alpha \mathfrak{R}(x)$, which are linear independent at all point $x \in \overline{D}$. $\mathbf{e}_\alpha(x)$ can be regards as the basis vectors of the tangential surface at this point. Meanwhile, the unit normal vector at the same point is

$$\mathbf{n} = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|}.$$

So $(\mathbf{e}_\alpha, \mathbf{n})$ constitute the covariant basis vectors at the point $\mathfrak{R}(x)$. We can further define contravariant basis vectors $(\mathbf{e}^\alpha, \mathbf{n})$ which satisfy the following relations

$$\mathbf{e}^\alpha \cdot \mathbf{e}_\beta = \delta^\alpha_\beta, \quad \mathbf{e}^\alpha \cdot \mathbf{n} = 0, \quad \mathbf{n} \cdot \mathbf{n} = 1,$$

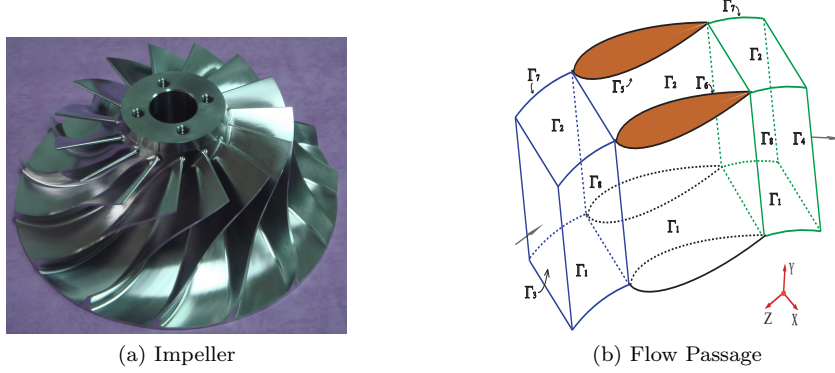


Figure 1: Impeller and Flow Passage

where δ_β^α denotes the Kronecker delta. It is well known that \mathbf{e}^α are also on the tangential plane to \mathfrak{S} at point $\mathfrak{R}(x)$.

The covariant components $a_{\alpha\beta}$ and contra-variant components $a^{\alpha\beta}$ of the metric tensor of \mathfrak{S} , the Christoffel symbols $\Gamma_{\beta\lambda}^*$, and the covariant components $b_{\alpha\beta}$ and mixed components b_α^β of the curvature tensor of \mathfrak{S} are defined as, respectively,

$$\begin{aligned} a_{\alpha\beta} &:= \mathbf{e}_\alpha \cdot \mathbf{e}_\beta, & a^{\alpha\beta} &= \mathbf{e}^\alpha \cdot \mathbf{e}^\beta, & \Gamma_{\beta\lambda}^* &:= \mathbf{e}^\alpha \cdot \partial_\beta \mathbf{e}_\lambda, \\ b_{\alpha\beta} &:= \mathbf{n} \cdot \partial_\beta \mathbf{e}_\alpha, & b_\alpha^\beta &= a^{\beta\sigma} b_{\alpha\sigma}. \end{aligned}$$

The area element along \mathfrak{S} is $\sqrt{a}dx$, where $a = \det(a_{\alpha\beta})$, $\sqrt{a} = |\mathbf{e}_1 \times \mathbf{e}_2|$.

Nest, we assume that the impeller is rotating along z-axis with angular velocity $\boldsymbol{\omega} = (0, 0, \omega)$. Let $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{k})$ be the cylindrical basis vectors established on the impeller and rotating with the impeller (see Appendix). Let constant N be the number of blade and $\varepsilon = \pi/N$. Then by rotating $\frac{2\pi}{N}$ degrees, one blade is rotated to the location of the next one. The flow passage Ω_ε of the impeller is the inner part of the boundary $\partial\Omega_\varepsilon = \Gamma_{in} \cup \Gamma_{out} \cup \Gamma_t \cup \Gamma_b \cup \mathfrak{S}_+ \cup \mathfrak{S}_-$. Further, the blade is $\mathfrak{S} = \mathfrak{R}(D)$, and an arbitrary point $\mathfrak{R}(x)$ on the blade \mathfrak{S} can be expressed as

$$\mathfrak{R}(x) = x^2 \mathbf{e}_r + x^2 \Theta(x) \mathbf{e}_\theta + x^1 \mathbf{k}, \quad (2.1)$$

where $x = (x^1, x^2) \in \bar{D}$ are Gauss coordinate system on \mathfrak{S} , and $\Theta \in C^2(D, R)$ is an smoothing enough function.

It is easy to prove that there exists a family of surfaces \mathfrak{S}_ξ , only depending on parameter ξ , which cover the domain Ω_ε by mapping $\mathfrak{R}(x; \xi) : D \rightarrow \mathfrak{S}_\xi$, where

$$\mathfrak{R}(x; \xi) = x^2 \mathbf{e}_r + x^2 (\varepsilon \xi + \Theta(x)) \mathbf{e}_\theta + x^1 \mathbf{k}. \quad (2.2)$$

It is easy to prove that the metric tensor $a_{\alpha\beta}$ of \mathfrak{S}_ξ is homogenous, nonsingular and independent of ξ , which is given as follows

$$\begin{aligned} a_{\alpha\beta} &= \frac{\partial \mathfrak{R}(x; \xi)}{\partial x^\alpha} \cdot \frac{\partial \mathfrak{R}(x; \xi)}{\partial x^\beta} = \delta_{\alpha\beta} + r^2 \Theta_\alpha \Theta_\beta, & a^{\alpha\beta} a_{\beta\lambda} &= \delta_\lambda^\alpha, \\ a &= \det(a_{\alpha\beta}) = 1 + r^2 (\Theta_1^2 + \Theta_2^2) > 0. \end{aligned} \quad (2.3)$$

Let $(x^{1'}, x^{2'}, x^{3'}) = (r, \theta, z)$, the corresponding metric tensor in R^3 are $(g_{1'1'} = 1, g_{2'2'} = r^2, g_{3'3'} = 1, g_{i'j'} = 0 \forall i' \neq j')$. According to rule of tensor transformation under coordinate transformation we have the following calculation formulae

$$g_{ij} = g_{i'j'} \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^{j'}}{\partial x^j}.$$

Substituting (2.3) into the above formula, the covariant and contra-variant components of the metric tensor of R^3 in the new curvilinear coordinate system are give by

$$\begin{cases} g_{\alpha\beta} = a_{\alpha\beta}, & g_{3\beta} = g_{\beta 3} = \varepsilon r^2 \Theta_\beta, & g_{33} = \varepsilon^2 r^2, & g = \det(g_{ij}) = \varepsilon^2 r^2, \\ g^{\alpha\beta} = \delta^{\alpha\beta}, & g^{3\beta} = g^{\beta 3} = -\varepsilon^{-1} \Theta_\beta, & g^{33} = \varepsilon^{-2} r^{-2} (1 + r^2 |\nabla \Theta|^2) = (r\varepsilon)^{-2} a, \end{cases} \quad (2.5)$$

where $|\nabla \Theta|^2 = \Theta_1^2 + \Theta_2^2$, and $\Theta_\alpha = \frac{\partial \Theta}{\partial x^\alpha}$.

Tensor calculations show that the following proposition is right (see Appendix),

Proposition 2.1. *let (e_α, \mathbf{n}) denote the basis vectors in new coordinate system (x, ξ) , and a vector $\mathbf{v} \in R^3$ can be wrote as $\mathbf{v} = v^\alpha e_\alpha + v^3 \mathbf{n}$. Further more, we have the following representation formula in new coordinate system,*

1. Angular velocity vector $\boldsymbol{\omega}$

$$\begin{cases} \boldsymbol{\omega} = \omega e_1 - \omega \varepsilon^{-1} \Theta_1 e_3 \\ \omega^1 = \omega, & \omega^2 = 0, & \omega^3 = -\omega \varepsilon^{-1} \Theta_1, \end{cases} \quad (2.6)$$

2. Coriolis Force

$$\begin{cases} \mathbf{C} = 2\boldsymbol{\omega} \times \mathbf{w} = C^i e_i, \\ C^1 = 0, & C^2 = -2\omega r \Pi(\mathbf{w}, \Theta), & C^3 = 2\omega \varepsilon^{-1} (r \Theta_2 \Pi(\mathbf{w}, \Theta) + \frac{w^2}{r}), \\ \Pi(\mathbf{w}, \Theta) = \varepsilon w^3 + w^\lambda \Theta_\lambda, \end{cases} \quad (2.7)$$

3. Unit normal vector to \mathfrak{S}

$$\mathbf{n} = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|} = n^i e_i, \quad n^\lambda = -r \Theta_\lambda / \sqrt{a}, \quad n^3 = (\varepsilon r)^{-1} \sqrt{a}, \quad (2.8)$$

4. Second fundamental form (curvature tensors for 2D manifold)

$$\begin{aligned} b_{11} &= \frac{1}{\sqrt{a}} (\Theta_2 (a_{11} - 1) + r \Theta_{11}), & b_{12} &= \frac{1}{\sqrt{a}} (\Theta_1 (a_{12} + r \Theta_{12}), \\ b_{22} &= \frac{1}{\sqrt{a}} (\Theta_2 (a_{22} + 1) + r \Theta_{22}), & b &= \det(b_{\alpha\beta}) = b_{11} b_{22} - b_{12}^2. \end{aligned} \quad (2.9)$$

5. Mean Curvature H and Gaussian Curvature K

$$\begin{cases} H = \frac{1}{2a\sqrt{a}} (\Theta_2 (a + a_{11} a_{22} + a_{11} - a_{22}) + r (a_{22} \Theta_{11} + a_{11} \Theta_{22} - 2a_{12} \Theta_{12})), \\ K = \frac{b}{a} = \det(b_{\alpha\beta}) / a. \end{cases} \quad (2.10)$$

It is obvious that each $\xi = \text{const}$ corresponds to a surface \mathfrak{S}_ξ which has the same geometry properties with \mathfrak{S} . It is well known that the geometry properties of \mathfrak{S} is completely determined by $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ in the following meaning.

Let \mathcal{O}^3 denote the set of all third-order orthogonal matrices, and that $\mathcal{O}_+^3 = \{Q \in \mathcal{O}^3, \det(Q) = 1\}$ denotes the subset of all proper third-order orthogonal matrices. Then $\mathbf{J}_+(x) = \mathbf{C} + Q \circ \mathbf{x}$ is a proper isometry of $R^3 \rightarrow R^3$, where $\mathbf{C} \in R^3, Q \in \mathcal{O}_+^3$. We have

Theorem 2.1. *Two immersions $\mathfrak{R}_1 \in C^1(D; R^3)$ and $\mathfrak{R}_2 \in C^1(D; R^3)$ share the same fundamental forms $(a_{\alpha\beta})$, $(b_{\alpha\beta})$ over an open connected subset $D \in R^3$ if and only if*

$$\mathfrak{R}_2 = \mathbf{J}_+ \circ \mathfrak{R}_1, \quad (2.11)$$

Furthermore, If two matrices fields $(a_{\alpha\beta}) \in C^2(D; S^2)$, $(b_{\alpha\beta}) \in C^2(D; S^2)$ satisfy Gauss-Codazzi equations in D , i.e.,

$$\begin{aligned} \partial_\beta \Gamma_{\alpha\sigma, \tau} - \partial_\sigma \Gamma_{\alpha\beta, \tau} + \Gamma_{\alpha\beta}^\mu \Gamma_{\sigma\tau, \mu} - \Gamma_{\alpha\sigma}^\mu \Gamma_{\beta\tau, \mu} &= b_{\alpha\sigma} b_{\beta\tau} - b_{\alpha\beta} b_{\sigma\tau}, \\ \partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + \Gamma_{\alpha\sigma}^\mu b_{\beta\mu} - \Gamma_{\alpha\beta}^\mu b_{\sigma\mu} &= 0, \end{aligned}$$

where $\Gamma_{\alpha\beta, \tau} = \frac{1}{2} (\partial_\alpha a_{\beta\tau} + \partial_\beta a_{\alpha\tau} - \partial_\tau a_{\alpha\beta})$, $\Gamma_{\alpha\beta}^\sigma = a^{\sigma\tau} \Gamma_{\alpha\beta, \tau}$. Then there exists an immersion $\mathfrak{R} \in C^3(D; R^3)$ such that

$$a_{\alpha\beta} = \partial_\alpha \mathfrak{R} \partial_\beta \mathfrak{R}, \quad b_{\alpha\beta} = \partial_{\alpha\beta}^2 \mathfrak{R} \cdot \left\{ \frac{\partial_1 \mathfrak{R} \times \partial_2 \mathfrak{R}}{|\partial_1 \mathfrak{R} \times \partial_2 \mathfrak{R}|} \right\}.$$

Because the surface \mathfrak{S}_ξ is obtained by a $\xi\varepsilon$ degree rotation of \mathfrak{S} , so by using theorem 2.1, we have the surface \mathfrak{S}_ξ have the same geometry properties with \mathfrak{S} for any $\xi \in [-1, 1]$, i.e., their corresponding geometric quantities $a_{\alpha\beta}, b_{\alpha\beta}, K, H, \dots$ are same.

In subsequent discussion, we will often employ the third fundamental tensor of \mathfrak{S} , i.e.,

$$c_{\alpha\beta} = a^{\lambda\sigma} b_{\alpha\lambda} b_{\beta\sigma}, \quad (2.12)$$

and inverse matrix $(\widehat{c}^{\alpha\beta}) = (c_{\alpha\beta})^{-1}, (\widehat{b}^{\alpha\beta}) = (b_{\alpha\beta})^{-1}$ satisfy the following relations,

$$\widehat{b}^{\alpha\beta} b_{\beta\lambda} = \delta_\lambda^\alpha, \quad \widehat{c}^{\alpha\beta} c_{\beta\lambda} = \delta_\lambda^\alpha. \quad (2.13)$$

Furthermore, let introduce permutation tensors in Euclid space R^3

$$\varepsilon_{ijk} = \begin{cases} \sqrt{g}, & \\ -\sqrt{g}, & \varepsilon_{ijk} = \begin{cases} \frac{1}{\sqrt{g}}, & (i,j,k) \text{ is even permutation of } (1,2,3), \\ -\frac{1}{\sqrt{g}}, & (i,j,k) \text{ is odd permutation of } (1,2,3), \\ 0, & \text{otherwise,} \end{cases} \end{cases} \quad (2.14)$$

where $g = \det(g_{ij})$. Similar permutation tensors on 2D manifold \mathfrak{S}

$$\varepsilon_{\alpha\beta} = \begin{cases} \sqrt{a}, & \\ -\sqrt{a}, & \varepsilon_{\alpha\beta} = \begin{cases} \frac{1}{\sqrt{a}}, & (\alpha, \beta) \text{ is even permutation of } (1,2), \\ -\frac{1}{\sqrt{a}}, & (\alpha, \beta) \text{ is odd permutation of } (1,2), \\ 0, & \text{otherwise.} \end{cases} \end{cases} \quad (2.15)$$

3 Rotating Navier-Stokes Equations in the New Coordinate System

we consider the rotating impeller with rotating angular velocity $\boldsymbol{\omega} = (0, 0, \omega)$. Under the rotating cylindrical coordinate established on the impeller, The motion of fluid in the flow passage is governed by the three-dimensional rotating Navier-Stokes equations, i.e.,

$$\begin{cases} \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{w}) = 0, \\ \rho \mathbf{a} = \text{div} \boldsymbol{\sigma} + \mathbf{f}, \\ \rho c_v \left(\frac{\partial T}{\partial t} + w^j \nabla_j T \right) - \text{div}(\kappa \text{grad} T) + p \text{div} \mathbf{w} - \Phi = h, \\ p = p(\rho, T), \end{cases} \quad (3.1)$$

where ρ the density of the fluid, w the velocity of the fluid, h the heat source, T the temperature, k the coefficient of heat conductivity, C_v specific heat at constant volume, and μ viscosity. Furthermore, the strain rate tensor, stress tensor, dissipative function and viscous tensor are respectively given by

$$\begin{aligned} e_{ij}(\mathbf{w}) &= \frac{1}{2}(\nabla_i w_j + \nabla_j w_i), & e^{ij}(\mathbf{w}) &= g^{ik} g^{jm} e_{km}(w) = \frac{1}{2}(\nabla^i w^j + \nabla^j w^i), \\ \sigma^{ij}(\mathbf{w}, p) &= A^{ijk m} e_{km}(w) - g^{ij} p, & A^{ijk m} &= \lambda g^{ij} g^{km} + \mu(g^{ik} g^{jm} + g^{im} g^{jk}) \\ \Phi &= A^{ijk m} e_{ij}(w) e_{km}(\mathbf{w}), & \lambda &= -\frac{2}{3}\mu, \end{aligned} \quad (3.2)$$

where g_{ij} , and g^{ij} are the covariant and contra-variant components of the metric tensor of three-dimensional Euclidian space in the curvilinear coordinate (x, ξ) define by (2.4), respectively. The covariant derivatives of velocity vector and Christoffel symbols are

$$\nabla_i w^j = \frac{\partial w^j}{\partial x^i} + \Gamma_{ik}^j w^k, \quad \nabla_i w_j = \frac{\partial w_j}{\partial x^i} - \Gamma_{ij}^k w_k, \quad \Gamma_{jk}^i = g^{il} \left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right). \quad (3.3)$$

The absolute acceleration of the fluid is given by

$$\begin{aligned} \mathbf{a} &= \frac{\partial \mathbf{w}}{\partial t} + (\mathbf{w} \nabla) \mathbf{w} + 2\boldsymbol{\omega} \times \mathbf{w} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}), \\ a^i &= \frac{\partial w^i}{\partial t} + w^j \nabla_j w^i + 2\varepsilon^{ijk} \omega_j w_k - \omega^2 r^i. \end{aligned} \quad (3.4)$$

where \mathbf{R} is the radius vector of the fluid particle. The flow passage occupied by the fluids is denoted by Ω_ε , which boundary $\partial\Omega_\varepsilon$ is the union of inflow boundary Γ_{in} , outflow boundary Γ_{out} , positive blade's surface \mathfrak{S}_+ , negative blade's surface \mathfrak{S}_- and top wall Γ_t and bottom wall Γ_b , i.e.,

$$\partial\Omega_\varepsilon = \Gamma = \Gamma_{in} \cup \Gamma_{out} \cup \mathfrak{S}_- \cup \mathfrak{S}_+ \cup \Gamma_t \cup \Gamma_b \quad (\text{Fig.2}) \quad (3.5).$$

The boundary conditions are given by

$$\begin{cases} \mathbf{w}|_{\mathfrak{S}_- \cup \mathfrak{S}_+} = \mathbf{0}, & \mathbf{w}|_{\Gamma_b} = \mathbf{0}, & \mathbf{w}|_{\Gamma_t} = \mathbf{0}, \\ \sigma^{ij}(\mathbf{w}, p) n_j|_{\Gamma_{in}} = g_{in}^i, & \sigma^{ij}(\mathbf{w}, p) n_j|_{\Gamma_{out}} = g_{out}^i, \\ \frac{\partial T}{\partial n} + \lambda(T - T_0) = 0, & \lambda \geq 0. \end{cases} \quad (3.6)$$

We also supply the initial condition

$$\mathbf{w}|_{t=0} = \mathbf{w}_0(x).$$

If the fluid is incompressible and flow is stationary, then the governing equations are

$$\begin{cases} \text{div} \mathbf{w} = 0, \\ (\mathbf{w} \nabla) \mathbf{w} + 2\boldsymbol{\omega} \times \mathbf{w} + \nabla p - \nu \text{div}(e(\mathbf{w})) = -(\boldsymbol{\omega})^2 \mathbf{R} + \mathbf{f}, \\ \mathbf{w}|_{\Gamma_0} = \mathbf{0}, \\ (-p\mathbf{n} + 2\nu e(\mathbf{w}))|_{\Gamma_{in}} = \mathbf{g}_{in}, \\ (-p\mathbf{n} + 2\nu e(\mathbf{w}))|_{\Gamma_{out}} = \mathbf{g}_{out}, \end{cases} \quad (3.7)$$

where $\Gamma_0 = \mathfrak{S}_+ \cup \mathfrak{S}_- \cup \Gamma_t \cup \Gamma_b$. For the polytropic ideal gas and flow is stationary, system (3.1) can be wrote as the conservation form

$$\begin{cases} \text{div}(\rho \mathbf{w}) = 0, \\ \text{div}(\rho \mathbf{w} \otimes \mathbf{w}) + 2\rho \boldsymbol{\omega} \times \mathbf{w} + R \nabla(\rho T) = \mu \Delta \mathbf{w} + (\lambda + \mu) \nabla \text{div} \mathbf{w} - \rho(\boldsymbol{\omega})^2 \mathbf{R}, \\ \text{div}[\rho(\frac{|\mathbf{w}|^2}{2} + c_v T + RT)\mathbf{w}] = \kappa \Delta T + \lambda \text{div}(\mathbf{w} \text{div} \mathbf{w}) + \mu \text{div}[\mathbf{w} \nabla \mathbf{w}] + \frac{\mu}{2} \Delta |\mathbf{w}|^2, \end{cases} \quad (3.8)$$

while for isentropic ideas gases, it turns

$$\begin{cases} \text{div}(\rho \mathbf{w}) = 0, \\ \text{div}(\rho \mathbf{w} \otimes \mathbf{w}) + 2\rho \boldsymbol{\omega} \times \mathbf{w} + \alpha \nabla(\rho^\gamma) = 2\mu \text{div}(e) + \lambda \nabla \text{div} \mathbf{w} - \rho(\boldsymbol{\omega})^2 \mathbf{R}, \end{cases} \quad (3.9)$$

where $\gamma > 1$ is the specific heat ratio and α a positive constant.

Furthermore, we give the expressions of the power $I(\mathfrak{S}, w(\mathfrak{S}))$ done by the impeller and global dissipative energy $J(\mathfrak{S}, \mathbf{w}(\mathfrak{S}))$, respectively

$$I(\mathfrak{S}, w(\mathfrak{S})) = \int \int_{\mathfrak{S}_- \cup \mathfrak{S}_+} \boldsymbol{\sigma} \cdot \mathbf{n} \cdot \mathbf{e}_\theta \omega r d\mathfrak{S}, \quad J(\mathfrak{S}, \mathbf{w}(\mathfrak{S})) = \int \int \int_{\Omega_\varepsilon} \Phi(\mathbf{w}) dV, \quad (3.10)$$

Under the new coordinate system (2.2), from the discussion in section 2, we know there exists mapping between the fixed domain $\Omega = D \times [-1, 1]$ and the flow passage Ω_ε . In the subsequent paragraph, we suppose $D \in \mathbb{R}^2$ is surrounded by four arcs \widehat{AB} , \widehat{CD} , \widehat{CB} , \widehat{DA} , and

$$\partial D = \gamma_0 \cup \gamma_1, \quad \gamma_0 = \widehat{AB} \cup \widehat{CD}, \quad \gamma_1 = \widehat{CB} \cup \widehat{DA}.$$

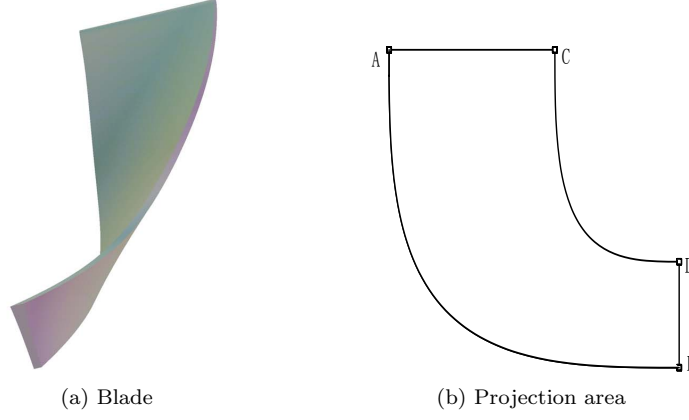


Figure 2: Blade and Projection area D of blade on the meridian.

There exist four positive functions $\gamma_0(z), \tilde{\gamma}_0(z), \gamma_1(z), \tilde{\gamma}_1(z)$ such that

$$\begin{aligned} r := x^2 = \gamma_0(x^1) = \gamma_0(z) & \quad \text{on } \widehat{AB}, & x^2 = \tilde{\gamma}_0(x^1) & \quad \text{on } \widehat{CD} \\ r := x^2 = \gamma_1(x^1) = \gamma_1(z) & \quad \text{on } \widehat{DA}, & x^2 = \tilde{\gamma}_1(x^1) & \quad \text{on } \widehat{BC}, \\ r_0 \leq \gamma_0(z) \leq r_1 & \quad \text{on } \widehat{AB}, & r_0 \leq \tilde{\gamma}_0(z) \leq r_1 & \quad \text{on } \widehat{CD}, \\ r_0 \leq \gamma_1(z) \leq r_1 & \quad \text{on } \widehat{DA}, & r_0 \leq \tilde{\gamma}_1(z) \leq r_1 & \quad \text{on } \widehat{BC}. \end{aligned} \quad (3.11)$$

The corresponding three-dimensional region is expressed as

$$\begin{aligned} \partial\Omega &= \tilde{\Gamma}_0 \cup \tilde{\Gamma}_1, & \tilde{\Gamma}_1 &= \tilde{\Gamma}_{out} \cup \tilde{\Gamma}_{in}, & \tilde{\Gamma}_0 &= \tilde{\Gamma}_b \cup \tilde{\Gamma}_t \cup \{\xi = 1\} \cup \{\xi = -1\}, \\ \tilde{\Gamma}_{in} &= \mathfrak{R}(\Gamma_{in}), \tilde{\Gamma}_{out} &= \mathfrak{R}(\Gamma_{out}), \tilde{\Gamma}_b &= \mathfrak{R}(\Gamma_b), \tilde{\Gamma}_t &= \mathfrak{R}(\Gamma_t), \end{aligned} \quad (3.12)$$

$$\partial D = \gamma_0 \cup \gamma_1, \quad \gamma_0 = (D \cap \tilde{\Gamma}_b) \cup (D \cap \tilde{\Gamma}_t), \quad \gamma_1 = (D \cup \tilde{\Gamma}_{out}) \cup (D \cup \tilde{\Gamma}_{in}), \quad (3.13)$$

where $\mathfrak{R}(\cdot)$ is defined by (2.1)

We introduce the following Sobolev space,

$$V(\Omega) = \{\mathbf{v} | \mathbf{v} \in H^1(\Omega)^3, \mathbf{v}|_{\tilde{\Gamma}_0} = 0\}, \quad H_\Gamma^1(\Omega) = \{q | q \in H^1(\Omega), q|_{\tilde{\Gamma}_0} = 0\}. \quad (3.14)$$

equipped with usual Sobolev norm $\|\cdot\|_{1,\Omega}$. Consider the variational formulation for Navier-Stokes problem (3.7) and (3.9)

$$\begin{cases} \text{Find } \mathbf{w} \in V(\Omega), p \in L^2(\Omega), \text{ such that} \\ a(\mathbf{w}, \mathbf{v}) + 2(\boldsymbol{\omega} \times \mathbf{w}, \mathbf{v}) + b(\mathbf{w}, \mathbf{w}, \mathbf{v}) - (p, \text{div} \mathbf{v}) = \langle \mathbf{F}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in V(\Omega), \\ (q, \text{div} \mathbf{w}) = 0, \quad \forall q \in L^2(\Omega), \end{cases} \quad (3.15)$$

and

$$\begin{cases} \text{Find } \mathbf{w} \in V(\Omega), \rho \in L^2(\Omega), \text{ such that} \\ a(\mathbf{w}, \mathbf{v}) + 2(\boldsymbol{\omega} \times \mathbf{w}, \mathbf{v}) + b(\rho \mathbf{w}, \mathbf{w}, \mathbf{v}) + (-p + \lambda \text{div} \mathbf{w}, \text{div} \mathbf{v}) = \langle \mathbf{F}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in V(\Omega), \\ (\nabla q, \rho \mathbf{w}) = \langle \rho \mathbf{w} \cdot \mathbf{n}, q \rangle|_{\Gamma_1}, \quad \forall q \in H_\Gamma^1(\Omega), \end{cases} \quad (3.16)$$

where

$$\begin{aligned} \langle \mathbf{F}, \mathbf{v} \rangle &= \langle \mathbf{f}, \mathbf{v} \rangle + \langle \tilde{\mathbf{g}}, \mathbf{v} \rangle_{\tilde{\Gamma}_1}, \quad \langle \tilde{\mathbf{g}}, \mathbf{v} \rangle = \langle \mathbf{g}_{in}, \mathbf{v} \rangle|_{\tilde{\Gamma}_{in}} + \langle \mathbf{g}_{out}, \mathbf{v} \rangle|_{\tilde{\Gamma}_{out}}, \\ a(\mathbf{w}, \mathbf{v}) &= \int_\Omega A^{ijk} e_{ij}(\mathbf{w}) e_{km}(\mathbf{v}) \sqrt{g} dx d\xi, \\ b(\mathbf{w}, \mathbf{w}, \mathbf{v}) &= \int_\Omega g_{km} w^j \nabla_j w^k v^m \sqrt{g} dx d\xi, \end{aligned} \quad (3.17)$$

In order to rewrite equations (3.7) and (3.9) in the new coordinate system, we have to consider covariant derivatives of the vector field. Therefore, we first give out the second kind of Christoffel symbols in the new coordinate system in terms of Θ (see Appendix), i.e.,

$$\begin{cases} \Gamma_{\beta\gamma}^\alpha = -r\delta_{2\alpha}\Theta_\beta\Theta_\gamma, & \Gamma_{3\beta}^\alpha = -\varepsilon r\delta_{2\alpha}\Theta_\beta, \\ \Gamma_{\alpha\beta}^3 = \varepsilon^{-1}r^{-1}(\delta_{2\alpha}\delta_\beta^\lambda + \delta_{2\beta}\delta_\alpha^\lambda)\Theta_\lambda + \varepsilon^{-1}\Theta_{\alpha\beta} + \varepsilon^{-1}r\Theta_2\Theta_\alpha\Theta_\beta, \\ \Gamma_{3\alpha}^3 = \Gamma_{\alpha 3}^3 = r^{-1}\delta_{2\alpha} + r\Theta_2\Theta_\alpha & \Gamma_{33}^\alpha = -\varepsilon^2r\delta_{2\alpha}, \quad \Gamma_{33}^3 = \varepsilon r\Theta_2, \end{cases} \quad (3.18)$$

then covariant derivatives of the velocity field is $\nabla_i w^j = \frac{\partial w^j}{\partial x^i} + \Gamma_{ik}^j w^k$, which can be specific expressed in the next lemma(see Appendix),

Lemma 3.1. *Under the new coordinate system (x^1, x^2, ξ) defined by (2.4), the covariant derivatives of the velocity field can be expressed as*

$$\begin{cases} \nabla_\alpha w^\beta = \frac{\partial w^\beta}{\partial x^\alpha} - r\delta_2^\beta \Theta_\alpha \Pi(\mathbf{w}, \Theta), \\ \nabla_\alpha w^3 = \frac{\partial w^3}{\partial x^\alpha} + \varepsilon^{-1}(x^2)^{-1}w^2\Theta_\alpha + \varepsilon^{-1}w^\beta\Theta_{\alpha\beta} + (\varepsilon x^2)^{-1}a_{2\alpha}\Pi(\mathbf{w}, \Theta), \\ \nabla_3 w^\alpha = \frac{\partial w^\alpha}{\partial \xi} - x^2\varepsilon\delta_{2\alpha}\Pi(\mathbf{w}, \Theta), \quad \nabla_3 w^3 = \frac{\partial w^3}{\partial \xi} + \frac{w^2}{x^2} + x^2\Theta_2\Pi(\mathbf{w}, \Theta), \\ \text{div} \mathbf{w} = \frac{\partial w^\alpha}{\partial x^\alpha} + \frac{w^2}{x^2} + \frac{\partial w^3}{\partial \xi}, \quad \Pi(\mathbf{w}, \Theta) = \varepsilon w^3 + w^\beta\Theta_\beta. \end{cases} \quad (3.19)$$

A simple calculation show that the strain tensor can be rewrite in the splitting form

$$e_{ij}(\mathbf{w}) = \phi_{ij}(\mathbf{w}) + \psi_{ij}(\mathbf{w}, \Theta), \quad (3.20)$$

where the first terms is independent of Θ , that is

$$\phi_{\alpha\beta}(\mathbf{w}) = \frac{1}{2}\left(\frac{\partial w^\alpha}{\partial x^\beta} + \frac{\partial w^\beta}{\partial x^\alpha}\right), \quad \phi_{3\alpha}(\mathbf{w}) = \frac{1}{2}\left(\frac{\partial w^\alpha}{\partial \xi} + \varepsilon^2 r^2 \frac{\partial w^3}{\partial x^\alpha}\right), \quad \phi_{33}(\mathbf{w}) = \varepsilon^2 r^2 \left(\frac{\partial w^3}{\partial \xi} + \frac{w^2}{r}\right). \quad (3.21)$$

While the second terms contains Θ , $\psi_{ij}(\mathbf{w}, \Theta) = \psi_{ij}^\lambda(\mathbf{w})\Theta_\lambda + \psi_{ij}^{\lambda\sigma}(\mathbf{w})\Theta_\lambda\Theta_\sigma + e_{ij}^*(\mathbf{w}, \Theta)$, where

$$\begin{cases} \psi_{\alpha\beta}^\lambda(\mathbf{w}) = \frac{1}{2}\varepsilon r^2 \left(\frac{\partial w^3}{\partial x^\alpha}\delta_\beta^\lambda + \frac{\partial w^3}{\partial x^\beta}\delta_\alpha^\lambda\right), \\ \psi_{3\alpha}^\lambda(\mathbf{w}) = \frac{1}{2}\varepsilon r^2 \left(\frac{\partial w^\lambda}{\partial x^\alpha} + \delta_\alpha^\lambda \left(\frac{\partial w^3}{\partial \xi} + \frac{2}{r}w^2\right)\right), \quad \psi_{33}^\lambda(\mathbf{w}) = \varepsilon r^2 \frac{\partial w^\lambda}{\partial \xi}, \\ \psi_{\alpha\beta}^{\lambda\sigma}(\mathbf{w}) = \frac{1}{2}r^2 \left(\frac{\partial w^\lambda}{\partial x^\alpha}\delta_{\beta\sigma} + \frac{\partial w^\lambda}{\partial x^\beta}\delta_{\sigma\alpha} + \frac{2}{r}w^2\delta_{\alpha\lambda}\delta_{\sigma\beta}\right), \\ \psi_{3\alpha}^{\lambda\sigma}(\mathbf{w}) = \frac{1}{2}r^2 \frac{\partial w^\lambda}{\partial \xi}\delta_{\alpha\sigma}, \quad \psi_{33}^{\lambda\sigma}(\mathbf{w}) = 0. \end{cases} \quad (3.22)$$

and

$$e_{\alpha\beta}^*(\mathbf{w}, \Theta) = \frac{1}{2}r^2 w^\sigma \partial_\sigma (\Theta_\alpha \Theta_\beta), \quad e_{3\alpha}^*(\mathbf{w}) = \frac{1}{2}\varepsilon r^2 w^\sigma \Theta_{\sigma\alpha}, \quad e_{33}^*(\mathbf{w}) = 0. \quad (3.23)$$

The proof is omitted.

The following notations are frequently used in the later,

$$\tilde{\Delta} = \frac{\partial^2}{\partial(x^1)^2} + \frac{\partial^2}{\partial(x^2)^2}, \quad \tilde{\nabla}_\alpha = \partial_\alpha = \frac{\partial}{\partial x^\alpha}, \quad \widetilde{\text{div}} w = \frac{\partial w^\alpha}{\partial x^\alpha}.$$

For the sake of simplicity, we just consider incompressible flow. Taking into account (3.18), (3.19), in the new coordinate system the Navier-Stokes equations can be written in the form,

Theorem 3.1. *Suppose that the blade surface is smooth enough, that is Θ is smooth enough, for example, $\Theta \in C^3(D)$, then the rotating Navier-Stokes equations in the new coordinate are given by*

$$\begin{cases} \frac{\partial w^\alpha}{\partial x^\alpha} + \frac{\partial w^3}{\partial \xi} + \frac{w^2}{r} = \frac{1}{r} \frac{\partial(rw^\alpha)}{\partial x^\alpha} + \frac{\partial w^3}{\partial \xi} = \widetilde{\text{div}}_2 w + \frac{\partial w^3}{\partial \xi} = 0, \\ \mathcal{N}^k(\mathbf{w}, p, \Theta) := -\nu \tilde{\Delta} w^k - \nu(r\varepsilon)^{-2} a \frac{\partial^2 w^k}{\partial \xi^2} - \nu P_j^{k3}(\Theta) \frac{\partial w^j}{\partial \xi} - 2\nu \varepsilon^{-1} \Theta_\beta \frac{\partial^2 w^k}{\partial \xi \partial x^\beta} \\ \quad - \nu P_j^{k\beta}(\Theta) \frac{\partial w^j}{\partial x^\beta} - \nu q_j^k(\Theta) w^j + g^{k\beta}(\Theta) \nabla_\beta p + g^{k3}(\Theta) \partial_\xi p \\ \quad + C^k(\mathbf{w}, \boldsymbol{\omega}) + N^k(\mathbf{w}, \mathbf{w}) = f^k, \quad \forall k = 1, 2, 3, \end{cases} \quad (3.24)$$

where $\mathbf{C}(\mathbf{w}, \boldsymbol{\omega})$ is Coriolis forces defined in (A.I.4), and

$$\begin{cases} N^\alpha(\mathbf{w}, \mathbf{w}) &= w^\beta \frac{\partial w^\alpha}{\partial x^\beta} + w^3 \frac{\partial w^\alpha}{\partial \xi} - r\delta_{2\alpha}\Pi(\mathbf{w}, \Theta)\Pi(\mathbf{w}, \Theta) = \frac{\partial(w^3 w^\alpha)}{\partial \xi} + \partial_\beta(w^\alpha w^\beta) \\ &+ r^{-1}w^2 w^\alpha - r\delta_{2\alpha}\Pi(\mathbf{w}, \Theta)\Pi(\mathbf{w}, \Theta), \\ N^3(\mathbf{w}, \mathbf{w}) &= \partial_\xi(w^3 w^3) + \partial_\beta(w^\beta w^3) + \varepsilon^{-1}w^\beta w^\lambda \Theta_{\beta\lambda} \\ &+ (r\varepsilon)^{-1}\Pi(\mathbf{w}, \Theta)(2w^2 + r^2\Theta_2\Pi(\mathbf{w}, \Theta)), \\ N^k(\mathbf{w}, \mathbf{w}) &= \frac{\partial(w^3 w^k)}{\partial \xi} + \partial_\beta(w^k w^\beta) + \pi_{ij}^k w^i w^j = \frac{\partial(w^3 w^k)}{\partial \xi} + B^k(\mathbf{w}, \mathbf{w}), \\ B^k(\mathbf{w}, \mathbf{w}) &= \partial_\beta(w^k w^\beta) + \pi_{ij}^k w^i w^j, \end{cases} \quad (3.25)$$

$$\begin{cases} P_\alpha^{\lambda\beta}(\Theta) &= \frac{1}{r}\delta_{\beta 2}, \quad P_3^{\lambda\beta}(\Theta) = -2r\varepsilon\delta_{2\lambda}\Theta_\beta, \\ P_\alpha^{3\beta}(\Theta) &= (r\varepsilon)^{-1}(\delta_{2\beta}\Theta_\alpha + 2r\Theta_{\alpha\beta}), \quad P_3^{3\beta} = \frac{2}{r}\delta_{\beta 2}, \\ P_\alpha^{\alpha 3}(\Theta) &= -[(r\varepsilon)^{-1}(\delta_{\alpha\lambda}\Theta_2 + 2\delta_{2\alpha}\Theta_\lambda) + \varepsilon^{-1}\delta_{\alpha\lambda}\tilde{\Delta}\Theta], \quad P_3^{\alpha 3} = -2r^{-1}\delta_{2\alpha}, \\ P_\sigma^{33}(\Theta) &= 2\varepsilon^{-2}(r^{-3}\delta_{2\sigma} - \Theta_\beta\Theta_{\beta\sigma}), \\ P_3^{33}(\Theta) &= \varepsilon^{-1}(r\Theta_2|\tilde{\nabla}\Theta|^2 - \tilde{\Delta}\Theta), \end{cases} \quad (3.26)$$

$$\begin{cases} q_\sigma^\alpha(\Theta) &= 2\delta_{2\alpha}[\delta_{2\sigma}|\tilde{\nabla}\Theta|^2 - a\Theta_2\Theta_\sigma + r\Theta_\sigma\tilde{\Delta}\Theta - r\Theta_\lambda\Theta_{\lambda\sigma}] - r^{-2}\delta_{2\sigma}\delta_{2\alpha}, \\ q_3^\alpha(\Theta) &= \delta_{2\alpha}(r\tilde{\Delta}\Theta - 2a\Theta_2)\varepsilon, \\ q_\sigma^3(\Theta) &= (r\varepsilon)^{-1}[r^{-1}(1 + a(a_{22} - 1))\Theta_\sigma + 2\Theta_{2\sigma}] + \varepsilon^{-1}\partial_\sigma\tilde{\Delta}\Theta, \\ q_3^3(\Theta) &= a\Theta_2\Theta_2, \end{cases} \quad (3.27)$$

$$\begin{cases} \Theta_\alpha = \frac{\partial\Theta}{\partial x^\alpha}, \quad \Theta_{\alpha\beta} = \frac{\partial^2\Theta}{\partial x^\alpha\partial x^\beta}, \quad \Pi(\mathbf{w}, \Theta) = \varepsilon w^3 + w^\lambda\Theta_\lambda, \\ \tilde{\Delta}\Theta = \Theta_{\alpha\alpha} = \Theta_{11} + \Theta_{22}, \quad |\tilde{\nabla}\Theta|^2 = \Theta_1^2 + \Theta_2^2, \end{cases} \quad (3.28)$$

and

$$\begin{cases} \pi_{\lambda\sigma}^\alpha(\Theta) &= -r\delta_{\alpha 2}\Theta_\lambda\Theta_\sigma + r^{-1}\delta_{\lambda 2}\delta_{\alpha\sigma}, \\ \pi_{\lambda 3}^\alpha(\Theta) &= \pi_{3\lambda}^\alpha(\Theta) = -r\varepsilon\delta_{2\alpha}\Theta_\lambda, \quad \pi_{33}^\alpha = -r\varepsilon^2\delta_{2\alpha}, \\ \pi_{\lambda\sigma}^3(\Theta) &= \varepsilon^{-1}\Theta_\lambda\Theta_\sigma + (r\varepsilon)^{-1}\Theta_\lambda(\delta_{2\sigma} + a_{2\sigma}), \\ \pi_{3\lambda}^3(\Theta) &= \pi_{\lambda 3}^3(\Theta) = r^{-1}a_{2\lambda} + r^{-1}\delta_{2\lambda}, \quad \pi_{33}^3 = r\varepsilon\Theta_2. \end{cases} \quad (3.29)$$

Proof: The Proof see Appendix. \square

Let introduce the inner product in the Sobolev space $V(\Omega)$ or $V(D)$

$$\begin{cases} (\mathbf{w}, \mathbf{v})_\Omega &= \int_\Omega [g_{ij}w^i v^j \sqrt{g} dx d\xi] = \int_\Omega [a_{\alpha\beta}w^\alpha v^\beta + r^2\varepsilon\Theta_\beta(w^3 v^\beta + w^\beta v^3) + r^2\varepsilon^2 w^3 v^3] r\varepsilon dx d\xi, \\ (\mathbf{w}, \mathbf{v})_D &= \int_D [a_{\alpha\beta}w^\alpha v^\beta + r^2\varepsilon\Theta_\beta(w^3 v^\beta + w^\beta v^3) + r^2\varepsilon^2 w^3 v^3] r\varepsilon dx, \end{cases} \quad (3.30)$$

The subscript “D” will be omitted if there is no misunderstanding.

Next we consider the variational formulation for (3.24) in the new coordinate system. Taking into account (2.5), let set

$$\begin{aligned} A(\mathbf{w}, \mathbf{v}) &= (g_{ij}\mathcal{N}^i(\mathbf{w}, p, \Theta), v^j) = (g_{\alpha\beta}\mathcal{N}^\alpha + \varepsilon r^2\Theta_\beta\mathcal{N}^3, v^\beta) + (\varepsilon r^2\Theta_\beta\mathcal{N}^\beta + e^2 r^2\mathcal{N}^3, v^3) \\ &= (A_m(\mathbf{w}, \Theta), v^m), \end{aligned} \quad (3.31)$$

By using index reduction, lift and descent of tensor, we get

$$\begin{aligned} g_{mk}g^{k\beta} &= \delta_m^\beta, \quad g_{mk}g^{k3} = \delta_m^3, \quad C_m(\mathbf{w}, \boldsymbol{\omega}) = g_{mk}C^k(\mathbf{w}, \boldsymbol{\omega}), \\ P_{mj}^l(\Theta) &= g_{mk}P_j^{kl}(\Theta), \quad q_{mj}(\Theta) = g_{mk}q_j^k(\Theta), \quad B_m(\mathbf{w}, \mathbf{w}) = g_{mk}B^k(\mathbf{w}, \mathbf{w}). \end{aligned} \quad (3.32)$$

Let adopt the notations

$$\begin{cases} E_m(\mathbf{w}) &= g_{mk}\tilde{\Delta}w^k = \partial_\lambda(g_{mk}\partial_\lambda w^k) - \partial_\lambda g_{mk}\partial_\lambda w^k, \\ N_m(\mathbf{w}, \mathbf{w}) &= g_{m\alpha}N^\alpha(\mathbf{w}, \mathbf{w}) + g_{m3}N^3(\mathbf{w}, \mathbf{w}) \\ &= g_{mk}\left(\frac{\partial(w^3 w^k)}{\partial \xi} + \partial_\beta(w^k w^\beta) + \pi_{ij}^k w^i w^j\right) = g_{mk}\frac{\partial(w^3 w^k)}{\partial \xi} + B_m(\mathbf{w}, \mathbf{w}), \\ B_m(\mathbf{w}, \mathbf{w}) &= g_{mk}(\partial_\beta(w^k w^\beta) + \pi_{ij}^k w^i w^j), \end{cases} \quad (3.33)$$

Then

$$\left\{ \begin{array}{l} A_m(\mathbf{w}, \Theta) = -\nu E_m(\mathbf{w}) - \nu(r\varepsilon)^{-2} a g_{mk} \frac{\partial^2 w^k}{\partial \xi^2} - \nu P_{mj}^3(\Theta) \frac{\partial w^j}{\partial \xi} - 2\nu \varepsilon^{-1} \Theta_\beta g_{mk} \frac{\partial^2 w^k}{\partial \xi \partial x^\beta} \\ \quad - \nu P_{mj}^\beta(\Theta) \frac{\partial w^j}{\partial x^\beta} - \nu q_{mj}(\Theta) w^j + \delta_m^\beta \nabla_\beta p + \delta_m^3 \partial_\xi p \\ \quad + C_m(\mathbf{w}, \omega) + B_m(\mathbf{w}, \mathbf{w}) + g_{mk} \partial_\xi (w^3 w^k) = f_m, \end{array} \right. \quad (3.34)$$

Remark 3.1. Obviously we have

$$\begin{aligned} (C(\mathbf{w}, \omega), \mathbf{w}) &= C_\beta(\mathbf{w}, \omega) w^\beta + C_3(\mathbf{w}, \omega) w^3 = 2r\omega(w^2 \Theta_\beta - \delta_{2\beta} \Pi(w, \Theta)) w^\beta + 2r\varepsilon \omega w^2 w^3 \\ &= 2r\omega(w^2 w^\beta \Theta_\beta - w^2(\varepsilon w^3 + w^\lambda \Theta_\lambda)) + 2r\varepsilon \omega w^2 w^3 = 0, \end{aligned} \quad (3.35)$$

which is coincide with $2\omega \times \mathbf{w} \cdot \mathbf{w} = 0$.

Since

$$\begin{aligned} -\nu g_{mk} \tilde{\Delta} w^k v^m &= -\nu g_{mk} v^m \partial_\lambda \partial_\lambda w^k \\ &= -\partial_\lambda (\nu g_{mk} v^m \partial_\lambda w^k) + \nu g_{mk} \partial_\lambda w^k \partial_\lambda v^m + \nu \partial_\lambda g_{mk} \partial_\lambda w^k v^m, \\ \int_{\xi=-1}^{\xi=1} \int_D [-\nu g_{mk} \tilde{\Delta} w^k v^m + \delta_m^\beta \partial_\beta p v^m] dx d\xi \\ &= \int_{\xi=-1}^{\xi=1} \int_D [-\partial_\lambda (\nu g_{mk} v^m \partial_\lambda w^k) + \partial_\beta (v^\beta p) + \nu g_{mk} \partial_\lambda w^k \partial_\lambda v^m + \nu \partial_\lambda g_{mk} \partial_\lambda w^k v^m - p \partial_\beta v^\beta] dx d\xi \\ &= \int_{\xi=-1}^{\xi=1} \int_{\partial D} [-\nu g_{mk} v^m \partial_\lambda w^k n_\lambda + p n_\beta v^\beta] ds d\xi \\ &\quad + \int_{\xi=-1}^{\xi=1} \int_D [\nu g_{mk} \partial_\lambda w^k \partial_\lambda v^m + \nu \partial_\lambda g_{mk} \partial_\lambda w^k v^m - p \partial_\beta v^\beta] dx d\xi, \\ \delta_m^\beta \partial_\beta p v^m &= \partial_\beta (v^\beta p) - p \partial_\beta v^\beta, \end{aligned}$$

where \mathbf{n} is unite normal vector to ∂D , then

$$\begin{aligned} \int_{\xi=-1}^{\xi=1} \int_D [-\nu g_{mk} \tilde{\Delta} w^k v^m + \delta_m^\beta \partial_\beta p v^m] dx d\xi &= \int_{\xi=-1}^{\xi=1} \int_{\partial D} [\sigma_{\mathbf{n}m}(\mathbf{w}, p) v^m] ds d\xi \\ &\quad + \int_{\xi=-1}^{\xi=1} \int_D [\nu g_{mk} \partial_\lambda w^k \partial_\lambda v^m + \nu \partial_\lambda g_{mk} \partial_\lambda w^k v^m - p \partial_\beta v^\beta] dx d\xi, \end{aligned} \quad (3.36)$$

where $\sigma_{\mathbf{n}m}(\mathbf{w}, p) = (-\nu g_{mk} \partial_\beta w^k + p \delta_{m\beta}) n_\beta$.

Recall the bilinear form and trilinear form on $V(\Omega)$

$$\left\{ \begin{array}{l} a_0(\mathbf{w}, \mathbf{v}) = \int_\Omega (\nu g_{mk} \partial_\lambda w^k \partial_\lambda v^m) dx d\xi, \\ b(\mathbf{w}, \mathbf{u}, \mathbf{v}) = \int_\Omega B_m(\mathbf{w}, \mathbf{w}) v^m \sqrt{g} dx d\xi = \int_\Omega g_{mk} [\partial_\lambda (w^k u^\lambda) + \pi_{ij}^k w^i u^j] v^m dx d\xi. \end{array} \right. \quad (3.37)$$

If the following boundary conditions are satisfied

$$\left\{ \begin{array}{l} \mathbf{w}|_{\Gamma_s} = 0, \quad \text{on } \Gamma_s = \Gamma_t \cup \Gamma_b \cup \Gamma|_{\xi=\pm 1}, \\ \sigma_{\mathbf{n}}(\mathbf{w}, p)|_{\Gamma_1} = \mathbf{h}, \quad \text{On } \Gamma_1 = \Gamma_{in} \cup \Gamma_{out}, \end{array} \right. \quad (3.38)$$

then the variational formulation for (3.24) is given by

$$\left\{ \begin{array}{l} \text{Find } \mathbf{w} \in V(\Omega), p \in L_0^2(\Omega), \text{ such that} \\ a_0(\mathbf{w}, \mathbf{v}) + (C(\mathbf{w}, \omega), \mathbf{v}) + b(\mathbf{w}, \mathbf{w}, \mathbf{v}) - (p, \partial_\alpha v^\alpha) + (\frac{\partial p}{\partial \xi}, v^3) - \nu((r\varepsilon)^{-2} a g_{mk} \frac{\partial^2 w^k}{\partial \xi^2}, v^m) \\ \quad + (\partial_\xi \Phi_m(\mathbf{w}, \Theta), v^m) + \nu(\tilde{P}_{mj}^\beta(\Theta)) \partial_\beta w^j, v^m - \nu(q_{mj}(\Theta) w^j, v^m) \\ \quad = (\mathbf{f}, \mathbf{v}) + \langle \mathbf{h}, \mathbf{v} \rangle := \langle \mathbf{F}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in V(\Omega) \\ (\frac{\partial w^\alpha}{\partial x^\alpha} + \frac{w^2}{r} + \frac{\partial w^3}{\partial \xi}, q) = 0, \quad \forall q \in L^2(\Omega), \end{array} \right. \quad (3.39)$$

where

$$\begin{cases} \Phi_m(\mathbf{w}, \Theta) = -\nu(P_{mj}^3(\Theta)w^j + 2\varepsilon^{-1}\Theta_\beta g_{mk} \frac{\partial w^k}{\partial x^\beta}) + g_{mk}(w^3 w^k), \\ \tilde{P}_{mj}^\beta(\Theta) = (\partial_\beta g_{mj} - P_{mj}^\beta(\Theta)), \end{cases} \quad (3.40)$$

4 The equations For the average velocity along the Rotating Direction

We define the average along the rotating direction for the function $\varphi(x^1, x^2, \xi)$ in the coordinate (x, ξ) in the domain $\Omega = D \times [-1, 1] \in R^3$

$$M(\varphi) = \frac{1}{2} \int_{-1}^1 \varphi(x, \xi) d\xi := \bar{\varphi}. \quad \forall \varphi(x, \xi) \in L^2(\Omega) \quad (4.1)$$

It is well known that the divergence of a vector \mathbf{w} can be written as under the coordinate (x^1, x^2, ξ)

$$\operatorname{div} \mathbf{w} = \frac{\partial w^\alpha}{\partial x^\alpha} + \frac{w^2}{r} + \frac{\partial w^3}{\partial \xi} = \frac{1}{r} \frac{\partial(rw^\alpha)}{\partial x^\alpha} + \frac{\partial w^3}{\partial \xi},$$

From this it yields

$$M(\operatorname{div} \mathbf{w}) = \frac{1}{2} \int_{-1}^1 \left[\frac{1}{r} \frac{\partial(rw^\alpha)}{\partial x^\alpha} + \frac{\partial w^3}{\partial \xi} \right] d\xi,$$

Since boundary conditions,

$$\begin{aligned} \frac{1}{2} \int_{-1}^1 \frac{\partial w^3}{\partial \xi} d\xi &= \frac{1}{2} (w^3|_{\xi=1} - w^3|_{\xi=-1}) = 0, \forall \mathbf{w} \in V(\Omega), \\ \int_{-1}^1 \frac{1}{r} \frac{\partial(rw^\alpha)}{\partial x^\alpha} d\xi &= \frac{1}{r} \frac{\partial}{\partial x^\alpha} (r \bar{w}^\alpha) = \frac{\partial \bar{w}^\alpha}{\partial x^\alpha} + \frac{\bar{w}^2}{r} := \widetilde{\operatorname{div}}_2(\bar{\mathbf{w}}), \end{aligned}$$

where

$$\widetilde{\operatorname{div}}_2(\mathbf{w}) = \frac{\partial w^\alpha}{\partial x^\alpha} + \frac{w^2}{r} = \frac{1}{r} \frac{\partial}{\partial x^\alpha} (rw^\alpha). \quad (4.2)$$

Therefore we assert

$$M(\operatorname{div} \mathbf{w}) = \widetilde{\operatorname{div}}_2(\bar{\mathbf{w}}). \quad (4.3)$$

and the incompressibility becomes

$$\widetilde{\operatorname{div}}_2(\overline{x^1, x^2 w}) = 0, \quad (4.4)$$

Taking into account the boundary conditions,

$$\mathbf{w}|_{\mathbb{S}_+ \cup \mathbb{S}_- \cup \gamma_t \cup \gamma_b} = \mathbf{0}, \quad \frac{\partial w^3}{\partial \xi}|_{\xi=\pm 1} = -\widetilde{\operatorname{div}}_2 \mathbf{w}|_{\xi=\pm 1} = 0. \quad (4.5)$$

we get

$$M(\partial_\xi \Phi(\mathbf{w}, \Theta)) = 0, \quad (4.6)$$

Let make notation $[\mathbf{w}] = \mathbf{w}|_{\xi=1} - \mathbf{w}|_{\xi=-1}$, $\bar{\mathbf{w}} = M\mathbf{w}$. Then average equations of Navier-Stokes equations are given by

$$\begin{cases} \widetilde{\operatorname{div}}_2 \bar{\mathbf{w}} = 0, \\ -\nu E_m(\bar{\mathbf{w}}) - \nu P_{mj}^\beta(\Theta) \frac{\partial \bar{w}^j}{\partial x^\beta} - \nu q_{mj}(\Theta) \bar{w}^j + \delta_m^\beta \nabla_\beta \bar{p} \\ \quad + C_m(\bar{\mathbf{w}}, \boldsymbol{\omega}) + M(B_m(\mathbf{w}, \mathbf{w})) = M(\mathbf{f})_m + \nu(r\varepsilon)^{-2} ag_{m\alpha} \left[\frac{\partial w^\alpha}{\partial \xi} \right] - \delta_m^3 [p], \end{cases} \quad (4.7)$$

Let $\mathbf{w}\mathbf{v} = a_{\lambda\sigma}w^\lambda v^\sigma + w^3v^3$, $\tilde{\mathbf{w}} = \mathbf{w} - \bar{\mathbf{w}}$, then, it is clear that

$$M(w^\lambda - \bar{w}^\lambda) = M\tilde{\mathbf{w}} = 0, \quad M(\tilde{\mathbf{w}}\bar{\mathbf{w}}) = 0, \quad (4.8)$$

Hence

$$\begin{cases} M(w^\lambda w^\sigma) = \bar{w}^\lambda \bar{w}^\sigma + M((\tilde{w}^\lambda)w^\sigma), \\ M(w^\lambda \frac{\partial w^k}{\partial x^\lambda}) = \bar{w}^\lambda \frac{\partial \bar{w}^k}{\partial x^\lambda} \\ \quad + M((\tilde{w}^\lambda) \frac{\partial w^k}{\partial x^\lambda}), \end{cases} \quad (4.9)$$

thus we conclude

$$\begin{aligned} M(B_m(\mathbf{w}, \mathbf{w})) &= M(g_{mk}(\frac{\partial w^\lambda w^k}{\partial x^\lambda} + \pi_{ij}^k(w^i w^j)), \\ M(B_m(\mathbf{w}, \mathbf{w})) &= B_m(\bar{\mathbf{w}}, \bar{\mathbf{w}}) + g_{mk}M(\partial_\lambda((\tilde{w}^\lambda)w^k) + \pi_{ij}^k(\tilde{w}^i)w^j), \end{aligned} \quad (4.10)$$

Finally, by virtue of

$$M(\tilde{\mathbf{w}}\mathbf{w}) = M(\tilde{\mathbf{w}}(\tilde{\mathbf{w}} + \bar{\mathbf{w}})) = M(\tilde{\mathbf{w}}\tilde{\mathbf{w}}),$$

it yields the reduced Navier-Stokes equations

$$\begin{cases} \operatorname{div}_2 \bar{\mathbf{w}} = 0, \\ -\nu E_m(\bar{\mathbf{w}}) - \nu P_{mj}^\beta(\Theta) \frac{\partial \bar{w}^j}{\partial x^\beta} - \nu q_{mj}(\Theta) \bar{w}^j + \delta_m^\beta \nabla_\beta \bar{p} \\ \quad + C_m(\bar{\mathbf{w}}, \omega) + (B_m(\bar{\mathbf{w}}, \bar{\mathbf{w}})) = M(\mathbf{f})_m + \nu(r\varepsilon)^{-2} ag_{m\alpha}[\frac{\partial w^\alpha}{\partial \xi}] - \delta_m^3[p] \\ \quad - g_{mk}M(\partial_\lambda((\tilde{w}^\lambda)\tilde{w}^k) + \pi_{ij}^k(\tilde{w}^i)\tilde{w}^j), \end{cases} \quad (4.11)$$

We define the Sobolev spaces

$$\begin{aligned} V(\Omega) &= \{\mathbf{u} \in \mathbf{h}^1(\Omega), \quad \mathbf{u} = 0, \text{ on } \Gamma_t \Gamma_b \cup \Gamma_+ \Gamma_-, \}, \\ V(D) &= \{\mathbf{u} \in \mathbf{h}^1, \mathbf{u} = 0, \text{ on } \gamma_0, \text{ see(3.13)}\}, \end{aligned}$$

By a similar manner as (3.39) the variational formulation for the reduced Navier-Stokes equations (4.11) is given as

$$\begin{cases} \text{Find } \bar{\mathbf{w}} \in V(D), \quad \bar{p} \in L^2(D) \text{ such that} \\ a_0(\bar{\mathbf{w}}, \mathbf{v}) + (\mathbf{C}(\bar{\mathbf{w}}, \omega), \mathbf{v}) - \nu(\tilde{P}_{mj}^\beta(\Theta) \partial_\beta \bar{w}^j + q_{mj} \bar{w}^j, v^m) + b(\bar{\mathbf{w}}, \bar{\mathbf{w}}, \mathbf{v}) - (\bar{p}, \partial_\alpha v^\alpha) \\ \quad = (-g_{mk}M(\partial_\lambda(\tilde{w} \tilde{w}^k) + \pi_{k,ij}(\tilde{w}^i \tilde{w}^j)), v^m) \\ \quad \quad + (\nu(r\varepsilon)^{-2} ag_{m\alpha}[\frac{\partial w^\alpha}{\partial \xi}] - \delta_{3m}[p], v^m) + (Mf_m, v^m), \quad \forall \mathbf{v} \in V(D) \\ (\operatorname{div}_2 \bar{\mathbf{w}}, q) = 0, \quad \forall q \in L^2(D), \end{cases} \quad (4.12)$$

where

$$\begin{cases} a_0(\mathbf{u}, \mathbf{v}) = (\nu g_{mk} \partial_\lambda u^k, \partial_\lambda v^m) = \int_D \nu g_{mk} \partial_\lambda u^k \partial_\lambda v^m dx, \\ b(\mathbf{u}, \mathbf{w}, \mathbf{v}) = (g_{mk}(\partial_\lambda(u^\lambda w^k) + \pi_{ij}^k(\Theta) u^i w^j), v^m), \end{cases} \quad (4.13)$$

5 The Equations for the Gâteaux Derivative of the solutions of NSE with Respect to the Shape of Boundary

In this section we consider the derivatives of the solution of NSE with respect to two dimensional manifold \mathfrak{S} which is a portion of the solid boundary of the flow in the channel in turbo-machinery.

Theorem 5.1. *Assume that Surface \mathfrak{S} is smooth enough, for example, $\Theta \in C^3(D)$, then there exists a Gâteaux derivatives ($\hat{w} := \frac{\mathcal{D}w}{\mathcal{D}\Theta}$, $\hat{p} := \frac{\mathcal{D}p}{\mathcal{D}\Theta}$) of the solutions (w, p) of Navier-Stokes equations (3.24) with respect to Θ satisfy the following linearized Navier-Stokes equations :*

$$\begin{cases} \operatorname{div} \hat{w} := \frac{\partial \hat{w}^\alpha}{\partial x^\alpha} + \frac{\partial \hat{w}^3}{\partial \xi} + \frac{\hat{w}^2}{r} = 0, \\ -\nu \tilde{\Delta} \hat{w}^k - \nu(r\varepsilon)^{-2} a \frac{\partial^2 \hat{w}^k}{\partial \xi^2} - \nu P_j^{k3}(\Theta) \frac{\partial \hat{w}^k}{\partial \xi} - 2\nu \varepsilon^{-1} \Theta_\beta \frac{\partial^2 \hat{w}^k}{\partial \xi \partial x^\beta} - \nu P_j^{k\beta}(\Theta) \frac{\partial \hat{w}^j}{\partial x^\beta} - \nu q_j^k(\Theta) \hat{w}^j \\ \quad + g^{k\beta} \partial_\beta \hat{p} + g^{k3} \partial_\xi \hat{p} + C^k(\hat{w}, \omega) + N^\alpha(w, \hat{w}) + N^\alpha(\hat{w}, w) + R^k(w, p, \Theta) = 0, \end{cases} \quad (5.1)$$

$$\begin{cases} \widehat{w} = 0, & \text{on } \Gamma_s \cap \{\xi = \xi_k\}, \\ \nu \frac{\partial \widehat{w}}{\partial n} - \widehat{p}n = 0, & \text{on } \Gamma_{in} \cap \Gamma_{out}, \end{cases} \quad (5.2)$$

where

$$\begin{cases} R^k(w, p, \Theta) \eta := -2\nu(r\varepsilon)^{-2} \Theta_\alpha \eta_\alpha \frac{\partial^2 w^k}{\partial \xi^2} - \nu \frac{DP_j^{k3}(\Theta)}{D\Theta} \eta \frac{\partial w^k}{\partial \xi} - 2\nu \varepsilon^{-1} \eta_\beta \frac{\partial^2 w^k}{\partial \xi \partial x^\beta} \\ - \nu \frac{DP_j^{k\beta}(\Theta)}{D\Theta} \eta \frac{\partial w^j}{\partial x^\beta} - \nu \frac{Dq_j^k(\Theta)}{D\Theta} \eta w^j + \frac{Dg^{k\beta}}{D\Theta} \eta \partial_\beta p + \frac{Dg^{k3}}{D\Theta} \eta \partial_\xi p \\ + 2\omega[-\Theta_\lambda \delta_2^k + r\varepsilon^{-1}(\delta_{\lambda 2} \Pi(w, \Theta) + \Theta_2 w^\lambda) \delta_3^k] \eta_\lambda + \frac{D\pi_{ij}^k(\Theta)}{D\Theta} \eta w^i w^j, \end{cases} \quad (5.3)$$

Proof: The proof see Appendix. \square

Associated variational formulation for (4.3) is given by

$$\begin{cases} \text{Find } \widehat{w} \in V(\Omega), \quad \widehat{p} \in L^2(\Omega) \quad \text{such that } \forall v \in V(\Omega) \\ a_0(\widehat{\mathbf{w}}, \mathbf{v}) + (\mathbf{C}(\widehat{w}, \omega), \mathbf{v}) + (\mathbf{L}(\widehat{w}, \Theta), \mathbf{v}) + b(\widehat{w}, \widehat{w}, v) - (\widehat{p}, \partial_\alpha v^\alpha) + (\frac{\partial \mathbf{T}(\widehat{w}, \widehat{w})}{\partial \xi}, \mathbf{v}) \\ = (f, v), \\ (\frac{\partial \widehat{w}^\alpha}{\partial x^\alpha} + \frac{\widehat{w}^2}{r} + \frac{\partial \widehat{w}^3}{\partial \xi}, q) = 0, \quad \forall \quad q \in L^2(\Omega), \end{cases} \quad (5.4)$$

where $b(\cdot, \cdot, \cdot)$ and $T(\cdot, \cdot)$ are respectively defined by (4.17) and (3.43).

6 2D-3C Navier-Stoke Equations on the 2D manifold \mathfrak{S}_ξ

As mentioned previously, for any $\xi = \text{const}$, there will correspond to a two dimensional surface \mathfrak{S}_ξ . On the other hand, the three components of coordinate (x, ξ) represent different meaning. the first two components x^α are variables on the tangent plane to the surface \mathfrak{S}_ξ , which describe the flow direction in the channel, and the third component ξ is transverse variable which describe transverse flow through different manifolds. Therefor the Navier-Stokes equations (3.24) can be decomposed into two parts, the first is the operator on the tangent plane to the surface \mathfrak{S}_ξ , which will be named “**Membrane Operator**”, meanwhile the second is the operator along the transverse direction, which is named “**Bending Operator**”. Proceeding from this thinking, under the new coordinate the Navier-Stokes equations (3.24) can be rewritten as

$$\begin{cases} \frac{\partial w^\alpha}{\partial x^\alpha} + \frac{\partial w^3}{\partial \xi} + \frac{w^2}{r} = \frac{1}{r} \frac{\partial(rw^\alpha)}{\partial x^\alpha} + \frac{\partial w^3}{\partial \xi} = \widetilde{\text{div}}_2 w + \frac{\partial w^3}{\partial \xi} = 0, \\ \mathcal{N}^i(\mathbf{w}, p, \Theta) := -\nu \widetilde{\Delta} w^i + \phi^{i\beta} \nabla_\beta p + C^i(\mathbf{w}, \omega) - \nu l^i(\mathbf{w}, \Theta) \\ + \frac{\partial}{\partial \xi}(\psi^i(\mathbf{w}, p, \Theta)) + B^i(\mathbf{w}, \mathbf{w}) = f^i, \end{cases} \quad (6.1)$$

where

$$\begin{cases} B^i(\mathbf{w}, \mathbf{w}) = \partial_\beta(w^i w^\beta) + \pi_{lj}^i w^l w^j, \quad \psi^i(\mathbf{w}, p, \Theta) = w^3 w^i + \eta^i p - \nu l_\xi^i(\mathbf{w}, \Theta), \\ \phi^{i\beta}(\Theta) = \delta^{i\beta} - \delta^{3i} \varepsilon^{-1} \delta^{\beta\sigma} \Theta_\sigma, \quad \eta^\alpha = -\varepsilon^{-1} \Theta_\alpha, \quad \eta^3 = (r\varepsilon)^{-2} a, \end{cases} \quad (6.2)$$

Let restrict the Navier-Stokes equations (6.2) on the surface \mathfrak{S}_{ξ_k} and adopt the Euler center difference to instead of the derivative with respect to ξ appearing in the bending operator \mathcal{N}_ξ . Then we introduce several abbreviation about jump operator and finite difference operators,

$$\begin{aligned} \mathbf{w}(k) &:= \mathbf{w}|_{\xi=\xi_k}, \quad [\mathbf{w}]_k := \mathbf{w}(k+1) - \mathbf{w}(k-1), \\ (\text{or}) [\mathbf{w}]_k &:= \mathbf{w}(k+1) - \mathbf{w}(k), \quad (\text{or}) [\mathbf{w}]_k := \mathbf{w}(k) - \mathbf{w}(k-1), \\ d_k^1(\mathbf{w}) &:= \frac{[\mathbf{w}]_k}{2\tau}, \quad d_k^2(\mathbf{w}) := -\frac{2\mathbf{w}(k)}{\tau^2} + \frac{1}{\tau^2} [\mathbf{w}]_k, \quad \widetilde{d}_k^2(\mathbf{w}) := \frac{1}{\tau^2} [\mathbf{w}]_k, \quad \tau = \xi_{k+1} - \xi_k, \end{aligned} \quad (6.3)$$

and the corresponding different quotient represent as

$$\begin{cases} \frac{\partial w^\alpha}{\partial \xi}|_{\xi_k} \cong d_k^1(w^\alpha) = \frac{1}{2\tau} (w^\alpha|_{\xi=\xi_{k+1}} - w^\alpha|_{\xi=\xi_{k-1}}) = \frac{1}{2\tau} [w^\alpha]_k, \\ d_k^2(w^\alpha) = \frac{\partial^2 w^\alpha}{\partial \xi^2}|_{\xi_k} \cong \frac{1}{\tau^2} (w^\alpha|_{\xi=\xi_{k+1}} - 2w^\alpha|_{\xi=\xi_k} + w^\alpha|_{\xi=\xi_{k-1}}) \end{cases} \quad (6.4)$$

Under this notations, we get

$$\begin{aligned}
\frac{\partial \psi^i}{\partial \xi} \big|_{\xi=\xi_k} &= \alpha_\tau w^i(k) + \frac{2\nu}{\tau \varepsilon} \Theta_\beta \frac{\partial w^i}{\partial x^\beta}(k) - \frac{\nu}{\tau} q_{\xi j}^i w^j(k) + \frac{1}{\tau} \eta^i p(k) + \frac{1}{\tau} w^3(k) w^i(k) \\
&\quad - \frac{1}{2} \alpha_\tau [w^i] - \frac{2\nu}{\tau \varepsilon} \Theta_\beta \frac{\partial w^i}{\partial x^\beta}(k-1) + \frac{\nu}{\tau} q_{\xi j}^i w^j(k-1) - \frac{1}{\tau} \eta^i p(k-1) - \frac{1}{\tau} w^3(k) w^i(k-1) \\
&= (\alpha_\tau \delta_j^i - \frac{\nu}{\tau} q_{\xi j}^i) w^j(k) + \frac{2\nu}{\tau \varepsilon} \Theta_\beta \frac{\partial w^i}{\partial x^\beta}(k) + \frac{1}{\tau} \eta^i p(k) + \frac{1}{\tau} w^3(k) w^i(k) + R_\tau(k-1), \\
R_\tau^i(k-1) &= -\frac{1}{2} \alpha_\tau [w^i] - \frac{2\nu}{\tau \varepsilon} \Theta_\beta \frac{\partial w^i}{\partial x^\beta}(k-1) + \frac{\nu}{\tau} q_{\xi j}^i w^j(k-1) - \frac{1}{\tau} \eta^i p(k-1) - \frac{1}{\tau} w^3(k) w^i(k-1),
\end{aligned}$$

where $\alpha_\tau = \frac{2\nu a}{r^2 \varepsilon^2 \tau^2}$

So we finally conclude that,

Theorem 6.1. *The 2D-3C Navier-Stokes problem restricted on a smooth 2D surface \mathfrak{S}_{ξ_k} is given by*

$$\begin{cases} \mathcal{N}^i(k) := -\nu \tilde{\Delta} w^i(k) + L_\tau^i(k) + C^i(\mathbf{w}(k), \boldsymbol{\omega}) + \phi^{i\beta} \nabla_\beta p(k) + \frac{1}{\tau} \eta^i p(k) \\ \quad + B^i(\mathbf{w}(k), \mathbf{w}(k)) + \frac{1}{\tau} w^3(k) w^i(k) = F_\tau^i(k), \\ \text{div}_2(\mathbf{w}(k)) = -d_k^1(w^3), \quad (\text{where } \text{div}_2 \mathbf{w} := \frac{1}{r} \partial_\alpha(r w^\alpha), d_\tau(\mathbf{w}) = d_k^1(w^3)), \end{cases} \quad (6.5)$$

with boundary conditions

$$\begin{cases} \mathbf{w}|_{\gamma_s} = 0, \quad \gamma_s = \Gamma_S \cap \{\xi = \pm 1\}, \\ \sigma \mathbf{n}(\mathbf{w}, p)|_{\gamma_{in}} = \mathbf{h}_{in}, \quad \gamma_{in} = \Gamma_{in} \cup \{\xi = \xi_k\} \\ \sigma \mathbf{n}(\mathbf{w}, p)|_{\gamma_{out}} = \mathbf{h}_{out}, \quad \gamma_{out} = \Gamma_{out} \cup \{\xi = \xi_k\} \end{cases} \quad (6.6)$$

where

$$\begin{cases} a = 1 + r^2 |\tilde{\nabla} \Theta|^2, \quad \sigma \mathbf{n}(\mathbf{w}, p) = -(\nu \frac{\partial w^\alpha}{\partial \mathbf{n}} - p n^\alpha) \mathbf{e}_\alpha - (\nu \frac{\partial w^3}{\partial \mathbf{n}}) \mathbf{e}_3, \\ L_\tau^i(k) = (\alpha_\tau \delta_j^i - \frac{\nu}{\tau} q_{\xi j}^i) w^j(k) - \nu l^i(w(k), \Theta) + \frac{2\nu}{\varepsilon \tau} \Theta_\beta \partial_\beta w^i(k) \\ \quad = (-\nu P_m^{i\beta} + \frac{2\nu}{\varepsilon \tau} \Theta_\beta \delta_m^i) \partial_\beta w^m + (-\nu q_j^i(\Theta) + \alpha_\tau \delta_j^i - \frac{\nu}{\tau} q_{\xi j}^i) w^j, \\ F_\tau^m(k) = f^m(k) + R_\tau^m(k-1), \end{cases} \quad (6.7)$$

Remark 6.1. *There exists a second order differential operator in $\partial_\xi \psi^m(w, p, \Theta)$,*

$$(r\varepsilon)^{-2} a \frac{\partial^2 w^m}{\partial \xi^2},$$

and the term $\alpha_\tau w^m(k)$ in (6.5) is obtained by using the different quotient of second order.

By a similar manner with (3.36), the equation satisfied by the covariant components of the Navier-Stokes equations are given as

$$\begin{aligned}
A_i(k) &:= g_{im} \mathcal{N}^m(k) = -\nu g_{im} \tilde{\Delta} w^m(k) + g_{im} L_\tau^m(k) + g_{im} C^m(\mathbf{w}(k), \boldsymbol{\omega}) + g_{im} \phi^{m\beta} \nabla_\beta p(k) \\
&\quad + \frac{1}{\tau} g_{im} \eta^m p(k) + g_{im} B^m(\mathbf{w}(k), \mathbf{w}(k)) + \frac{1}{\tau} g_{im} w^3(k) w^m(k) = g_{im} F_\tau^m(k),
\end{aligned}$$

Simple calculation shows

$$\begin{aligned}
g_{im} \phi^{m\beta} \nabla_\beta p(k) v^i &= g_{im} (\delta^{m\beta} - \delta^{3m} \varepsilon^{-1} \delta^{\beta\sigma} \Theta_\sigma) \partial_\beta p(k) v^i = (g_{i\beta} - g_{i3} \varepsilon^{-1} \Theta_\beta) \partial_\beta p(k) v^i \\
&= (g_{\alpha\beta} - g_{3\alpha} \varepsilon^{-1} \Theta_\beta) \partial_\beta p v^\alpha + (g_{3\beta} - g_{33} \varepsilon^{-1} \Theta_\beta) \partial_\beta p(k) v^3 = \delta_{\alpha\beta} \partial_\beta p v^\alpha \\
&= \partial_\alpha(p v^\alpha) - p \partial_\alpha v^\alpha, \\
g_{im} \eta^m v^i &= (-\varepsilon^{-1} \Theta_\alpha a_{\alpha\beta} + (r\varepsilon)^{-2} a g_{3\beta}) v^\beta + (-\varepsilon^{-1} \Theta_\alpha g_{3\alpha} + (r\varepsilon)^{-2} a g_{33}) v^3 = v^3, \\
g_{im} \tilde{\Delta} w^m v^i &= \partial_\lambda (g_{im} \partial_\lambda w^m v^i) - g_{im} \partial_\lambda w^m \partial_\lambda v^i - \partial_\lambda g_{im} \partial_\lambda w^m v^i,
\end{aligned}$$

$$\begin{aligned}
A_i(k) v^i &= -\nu \partial_\lambda (g_{im} \partial_\lambda w^m v^i) + \nu g_{im} \partial_\lambda w^m \partial_\lambda v^i + (\nu \partial_\lambda g_{im} \partial_\lambda w^m + g_{im} L_\tau^m(k)) v^i \\
&\quad + g_{im} C^m(w(k), \boldsymbol{\omega}) v^i + \partial_\alpha(p v^\alpha) - p \partial_\alpha v^\alpha + p v^3 \\
&\quad + g_{im} B^m(w(k), w(k)) v^i + \frac{1}{\tau} g_{im} w^3(k) w^m(k). \end{aligned} \quad (6.8)$$

By using Green's formula and boundary conditions, we get

$$\begin{aligned} \int_D [-\nu \partial_\lambda (g_{ij} \partial_\lambda (w^i) v^j) + \partial_\alpha (p v^\alpha)] dx &= \int_{\gamma_{in} \cup \gamma_{out}} (-\nu g_{ij} \frac{\partial w^i}{\partial n} v^j + p n_\alpha v^\alpha) ds \\ &= \int_{\gamma_{in} \cup \gamma_{out}} [g_{ij} \sigma_n^i(w, p) v^j] ds, \end{aligned}$$

where \mathbf{n} is normal vector to $\gamma_1 = \gamma_{in} \cup \gamma_{out}$, i.e.,

$$\begin{cases} \mathbf{n} = n^\alpha \mathbf{e}_\alpha + 0 \mathbf{e}_3, & n^3 = n_3 = 0, \\ \sigma \mathbf{n}(\mathbf{w}, p) = -\nu \frac{\partial \mathbf{w}}{\partial \mathbf{n}} + p \mathbf{n}. \end{cases} \quad (6.9)$$

Hence the variational formulation associated with (6.5) and (6.6) are expressed as

$$\begin{cases} \text{Find } \mathbf{w}(k) \in V(D), p \in L^2(D), \text{ such that} \\ a_0(\mathbf{w}, \mathbf{v}) + (\mathbf{L}(k), \mathbf{v}) + (\mathbf{C}(k), \mathbf{v}) + b(\mathbf{w}, \mathbf{w}, \mathbf{v}) + (p(k), v^3 - \partial_\alpha v^\alpha) \\ \quad = (\mathbf{G}_\tau(k), \mathbf{v}), \quad \forall \mathbf{v} \in V(D), \\ (\text{div}_2 \mathbf{w}, q) = (d_\tau(\mathbf{w}), q), \quad \forall q \in L^2(D), \end{cases} \quad (6.10)$$

where

$$\begin{cases} a_0(\mathbf{w}, \mathbf{v}) = \int_D \{ (\nu g_{ij} \tilde{\nabla}_\lambda w^i \tilde{\nabla}_\lambda v^j) dx \\ \quad = \int_D \nu [a_{\alpha\beta} \tilde{\nabla}_\lambda w^\alpha \tilde{\nabla}_\lambda v^\beta + r^2 \varepsilon \Theta_\beta (\tilde{\nabla}_\lambda w^\beta \tilde{\nabla}_\lambda v^3 + \tilde{\nabla}_\lambda w^3 \tilde{\nabla}_\lambda v^\beta) + r^2 \varepsilon^2 \tilde{\nabla}_\lambda w^3 \tilde{\nabla}_\lambda v^3] dx, \\ (\mathbf{C}(k), \mathbf{v}) = \int_D (g_{ij} C^i(k) v^j) dx = \int_D 2r\omega [(\Theta_\beta w^2 - \delta_{\beta 2} \Pi(w, \Theta)) v^\beta + \varepsilon w^2 v^3] dx, \\ (\mathbf{L}(w(k), \Theta), \mathbf{v}) = \int_D [(\nu \partial_\lambda g_{im} \partial_\lambda w^m + g_{im} L_\tau^m(k)) v^i] dx \\ \quad = (L_{ij}^\beta(\Theta) \partial_\beta w^j + L_{ij}(\Theta) w^j, v^i), \\ b(\mathbf{w}, \mathbf{u}, \mathbf{v}) = (\mathbf{B}(w, u), \mathbf{v}) = (g_{ij} B^i(w(k), w(k)) + g_{ij} \frac{1}{\tau} w^3(k) w^i(k), v^j) \\ \quad = ((\partial_\lambda (w^\lambda w^\alpha) + \Pi_{mk}^\alpha w^m w^k, a_{\alpha\beta} v^\beta + \varepsilon r^2 \Theta_\alpha v^3) \\ \quad \quad + ((\partial_\lambda (w^\lambda w^3) + \Pi_{mk}^3 w^m w^k), \varepsilon r^2 \Pi(v, \Theta)), \\ (\mathbf{G}_\tau(k), \mathbf{v}) = (\mathbf{f}_\tau(k), \mathbf{v}) + \langle \sigma_n(w, p), \mathbf{h} \rangle_{\gamma_1} \quad (\text{by (6.7)}) \end{cases} \quad (6.11)$$

where $\gamma_1 = \gamma_{in} \cup \gamma_{out}$,

$$\begin{cases} \tilde{\Delta} = \frac{\partial^2}{\partial x^1} + \frac{\partial^2}{\partial x^2}, & \text{div}_2 w = \frac{1}{r} \frac{\partial (r w^\alpha)}{\partial x^\alpha}, \\ V(D) := \{v | v \in \mathbf{h}^1(D), \quad v = 0 \text{ on } \gamma_s\}, \end{cases} \quad (6.12)$$

and

$$\begin{cases} L_{ij}^\beta(\Theta) = \frac{2\nu}{\varepsilon \tau} \Theta_\gamma g_{ij} - \nu g_{im} P_j^{m\beta}(\Theta) + \nu \partial_\beta g_{ij}, \\ L_{ij}(\Theta) = \alpha_\tau g_{ij} - \frac{\nu}{\tau} g_{im} q_{\xi j}^m - \nu g_{im} q_j^m, \\ \Pi_{ij}^3(\Theta) = \pi_{ij}^\alpha(\Theta) + \frac{1}{\tau} \delta_{3i} \delta_{\alpha j}, \quad \Pi_{ij}^3(\Theta) = \pi_{ij}^3(\Theta) + \frac{1}{\tau} \delta_{3i} \delta_{3j} \end{cases} \quad (6.13)$$

7 Pressure Correction Equation on the Blade Surface

Noting that we must give value of $[p]$ in the source term $\mathbf{F}_\tau(\mathbf{k})$ of equations (6.12), so the pressure on the surface \mathfrak{S} must be supplied. Therefore, we recall the Navier-Stokes equations in invariant form

$$\begin{cases} -\nu g^{jk} \nabla_j \nabla_k w^i + w^j \nabla_j w^i + 2\varepsilon^{ijk} g_{jm} g_{kl} \omega^m w^l + g^{ij} \nabla_j p = f^i, \\ \nabla_j w^j = 0, \end{cases} \quad (7.1)$$

Let take divergence ∇_i for (6.1)₁ and apply the identity $\nabla_k g_{ij} = \nabla_k g^{ij} = \nabla_k \varepsilon^{ijm} = 0$, then we have

$$-\nu g^{jk} \nabla_i \nabla_j \nabla_k w^i + \text{div}((\mathbf{w} \cdot \nabla) \mathbf{w}) + \text{div}(2\boldsymbol{\omega} \times \mathbf{w}) + g^{ij} \nabla_i \nabla_j p = \text{div} \mathbf{f}. \quad (7.2)$$

Because the Riemann curvature tensor vanishes in Euclidean space R^3 , therefore by exchanging the order of covariant derivatives, we have

$$-\nu g^{jk} \nabla_i \nabla_j \nabla_k w^i = g^{jk} \nabla_j \nabla_k (\nabla_i w^i) = 0, \quad (7.3)$$

In addition, a simple calculation shows that

$$\begin{cases} \operatorname{div}((\mathbf{w} \cdot \nabla) \mathbf{w}) = \nabla_i (w^j \nabla_j w^k) = \nabla_i w^j \nabla_j w^i + w^j \nabla_j \nabla_i w^i = \nabla_i w^j \nabla_j w^i \\ = \nabla_\alpha w^\beta \nabla_\beta w^\alpha + 2 \nabla_3 w^\beta \nabla_\beta w^3 + \nabla_3 w^3 \nabla_3 w^3. \end{cases} \quad (7.4)$$

From (2,7) we have,

$$\begin{aligned} \mathbf{C} &= 2\boldsymbol{\omega} \times \mathbf{w} = C^i \mathbf{e}_i, \quad C^1 = 0, \quad C^2 = -2\omega r \Pi(\mathbf{w}, \Theta), \quad C^3 = 2\omega \varepsilon^{-1} (r \Theta_2 \Pi(\mathbf{w}, \Theta) + \frac{w^2}{r}), \\ \operatorname{div} \mathbf{C} &= \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g} C^i}{\partial x^i} = C^i \partial_i \ln(er) + \partial_i C^i = r^{-1} C^2 + \frac{\partial}{\partial r} (-2r\omega \Pi(\mathbf{w}, \Theta)) \\ &= -2\omega r^{-1} (2r \Pi(\mathbf{w}, \Theta) + r^2 \frac{\partial}{\partial r} \Pi(\mathbf{w}, \Theta)) = -\frac{2\omega}{r} \frac{\partial}{\partial r} (r^2 \Pi(\mathbf{w}, \Theta)). \end{aligned} \quad (7.5)$$

On the other hand, the Laplace-Betrami operator can be expressed as

$$\begin{aligned} \Delta p &= \frac{1}{\sqrt{g}} \partial_j (g^{jk} \sqrt{g} \partial_k p) = (r\varepsilon)^{-1} [\partial_\alpha (r\varepsilon g^{\alpha\beta} \partial_\beta p) + (\partial_\xi (r\varepsilon g^{3\lambda} \partial_\lambda p) + \partial_\lambda (r\varepsilon g^{\lambda 3} \partial_\xi p)) \\ &\quad + \partial_\xi (r\varepsilon g^{33} \partial_\xi p)], \end{aligned}$$

By using (2.5), we claim that

$$\Delta p = \frac{1}{r\varepsilon} \left[\frac{\partial}{\partial x^\alpha} (r\varepsilon \frac{\partial p}{\partial x^\alpha}) - (2r\Theta_\lambda) \frac{\partial^2 p}{\partial x^\lambda \partial \xi} - (\Theta_2 + r\tilde{\Delta}\Theta) \frac{\partial p}{\partial \xi} + (r\varepsilon)^{-1} a \frac{\partial^2 p}{\partial \xi^2} \right], \quad (7.6)$$

Assume that centrifugation force is the only exterior force, that is

$$\mathbf{f} = -\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{R} = -\omega^2 \mathbf{R},$$

then

$$\nabla_i f^i = -\omega^2 \nabla_i r^i = \omega^2 (\nabla_\alpha r^\alpha + \nabla_3 r^3).$$

On the other hand, we have

$$\begin{aligned} \mathbf{R} &= r \mathbf{e}_r = r(\mathbf{e}_2 - \varepsilon^{-1} \Theta_2 \mathbf{e}_3), \quad r^2 = r, \quad r^1 = 0, \quad r^3 = -\varepsilon^{-1} r \Theta_2, \\ \nabla_i r^i &= \frac{\partial r^\alpha}{\partial x^\alpha} - r \Theta_2 \Pi(w, \Theta) + \frac{\partial r^3}{\partial \xi} + \frac{r^2}{r} + r \Theta_2 \Pi(w, \Theta) = 1 + 0 + 1 = 2. \\ \operatorname{div}(\mathbf{f}) &= -2|\boldsymbol{\omega}|^2. \end{aligned}$$

Summing up the above conclusions, we get

$$\begin{cases} \frac{1}{r\varepsilon} \left[\frac{\partial}{\partial x^\alpha} (r\varepsilon \frac{\partial p}{\partial x^\alpha}) - \frac{\partial}{\partial \xi} (2r\Theta_\lambda \frac{\partial p}{\partial x^\lambda} + (\Theta_2 + r\tilde{\Delta}\Theta)p) + (r\varepsilon)^{-1} a \frac{\partial^2 p}{\partial \xi^2} \right] \\ \quad + \nabla_\alpha w^\beta \nabla_\beta w^\alpha + 2 \nabla_3 w^\beta \nabla_\beta w^3 + \nabla_3 w^3 \nabla_3 w^3 \\ \quad - \frac{2\omega}{r} \frac{\partial}{\partial r} (r^2 \Pi(w, \Theta)) = -2|\boldsymbol{\omega}|^2. \end{cases} \quad (7.7)$$

Next we consider the restriction of equation (7.7) on any surface \mathfrak{S}_{ξ_k} . Noting that the Laplace-Betrami operator of pressure p in the new curvilinear coordinate system (x^α, ξ) can be split as the sum of two operators, membrane operator on tangent space and the bending operator along the rotational direction

$$\begin{cases} -\Delta p = -\Delta_m p - \Delta_b p, \\ -\Delta_m p = -\frac{1}{r\varepsilon} \frac{\partial}{\partial x^\alpha} (r\varepsilon \frac{\partial p}{\partial x^\alpha}) = -\frac{1}{r} \frac{\partial}{\partial x^\alpha} (r \frac{\partial p}{\partial x^\alpha}), \\ -\Delta_b p = -\frac{1}{r\varepsilon} [(r\varepsilon)^{-1} a \frac{\partial^2 p}{\partial \xi^2} + \frac{\partial}{\partial \xi} (2r\Theta_\lambda \frac{\partial p}{\partial x^\lambda} + (\Theta_2 + r\tilde{\Delta}\Theta)p)]. \end{cases} \quad (7.8)$$

We approximate the derivatives with respect to rotational variable in (7.7) by the difference quotients defined by (6.5), and then restricted it on the \mathfrak{S}_{ξ_k} , finally we get

$$\begin{cases} -\frac{1}{r} \frac{\partial}{\partial x^\alpha} (r \frac{\partial p_k}{\partial x^\alpha}) + \alpha_\tau p_k = f_k(\tau), \\ p|_{\gamma_{in}} = p_0, \\ \partial_n p = \tilde{f}_n, \text{ other boundaries.} \end{cases} \quad (7.9)$$

where

$$\begin{cases} \alpha_\tau(x) := \frac{a}{\tau^2 r \varepsilon} > \frac{1}{r \varepsilon \tau^2}, \quad \forall x \in \overline{D} \\ f_k(\tau) = -2r \Theta_\lambda d_k^1(\frac{\partial p}{\partial x^\lambda}) - (\Theta_2 + r \tilde{\Delta} \Theta) d_k^1(p) + \frac{a}{r \varepsilon} \tilde{d}_k^2(p) \\ \quad - 2\omega r^{-1} \frac{\partial}{\partial r} (r^2 \Pi(w(k), \Theta)) - 2|\omega|^2 \\ \quad + \nabla_\alpha w^\beta(k) \nabla_\beta w^\alpha(k) + 2\tilde{\nabla}_3 w^\beta(k) \nabla_\beta w^3(k) + \tilde{\nabla}_3 w^3(k) \tilde{\nabla}_3 w^3(k), \\ \tilde{\nabla}_3 w^\lambda := d_k^1(w^\lambda) - r \varepsilon \delta_{2\lambda} \Pi(w(k), \Theta), \\ \tilde{\nabla}_3 w^3 := d_k^1(w^3) + r^{-1} w^2(k) + r \Theta_2 \Pi(w(k), \Theta). \end{cases} \quad (7.10)$$

Specially, we consider the restriction of (7.7) on the surface $\mathfrak{S}_{\pm 1}$. When $\xi = \pm 1$, we have

$$\begin{aligned} \mathbf{w}|_{\xi=\pm 1} &= 0, \quad \partial_\alpha \mathbf{w} = 0, \quad \nabla_\alpha w^i = 0, \\ \nabla_3 w^3(\pm 1) &= \frac{\partial w^3}{\partial \xi}(\pm 1) = -(\partial_\alpha w^\alpha + r^{-1} w^2)(\pm 1) = 0, \\ \Pi(w, \Theta) &= \frac{\partial \Pi(w, \Theta)}{\partial r} = 0, \text{ (see (3.19)).} \end{aligned}$$

At present, equation (7.9) becomes

$$\begin{cases} -\frac{1}{r} \frac{\partial}{\partial x^\alpha} (r \frac{\partial p}{\partial x^\alpha}) \\ = -\frac{\partial}{\partial \xi} (2r \Theta_\lambda \frac{\partial p}{\partial x^\lambda} + (\Theta_2 + r \tilde{\Delta} \Theta) p) + (r \varepsilon)^{-1} a \frac{\partial^2 p}{\partial \xi^2} - 2(\omega)^2, \end{cases} \quad (7.11)$$

Furthermore, we replace the derivatives $\frac{\partial w}{\partial \xi}$ by difference quotient and apply the boundary condition $w|_{\xi=\pm 1} = 0$, then

$$\begin{aligned} \frac{\partial \mathbf{w}}{\partial \xi}|_{\xi=-1} &= \frac{1}{\tau} (\mathbf{w}|_{\xi=-1+\tau} - \mathbf{w}|_{\xi=-1}) = \frac{1}{\tau} (\mathbf{w}|_{\xi=-1+\tau}), \\ \frac{\partial^2 \mathbf{w}}{\partial \xi^2}|_{\xi=-1} &= \frac{1}{\tau^2} (\mathbf{w}|_{\xi=-1+2\tau} - 2\mathbf{w}|_{\xi=-1+\tau}), \end{aligned} \quad (7.12)$$

Borrowing the notations in (6.3), when $\xi = \pm 1$,

$$\begin{cases} d_{-1}(\mathbf{w}) = \frac{\partial \mathbf{w}}{\partial \xi}(-1) = \frac{1}{\tau} \mathbf{w}(-1 + \tau), \quad (w(-1) = 0), \\ d_{-1}^2(\mathbf{w}) = \frac{\partial^2 \mathbf{w}}{\partial \xi^2}(-1) = \frac{1}{\tau^2} (\mathbf{w}(-1 + 2\tau) - 2\mathbf{w}(-1 + \tau)), \quad (\mathbf{w}(-1) = 0), \end{cases} \quad (7.13)$$

Similarly, when $\xi = 1$,

$$\begin{cases} d_1(\mathbf{w}) = \frac{\partial \mathbf{w}}{\partial \xi}(1) = -\frac{1}{\tau} \mathbf{w}(1 - \tau), \quad (\mathbf{w}(1) = 0), \\ d_1^2(\mathbf{w}) = \frac{\partial^2 \mathbf{w}}{\partial \xi^2}(1) = \frac{1}{\tau^2} (\mathbf{w}(1 - 2\tau) - 2\mathbf{w}(1 - \tau)), \quad (\mathbf{w}(1) = 0), \end{cases} \quad (7.14)$$

In addition, Owing to $\xi = -1$ and $\xi = 1$ are both sides of blade, hence

$$p(-1 - \tau) \simeq p(1), \quad p(1 + \tau) = p(-1). \quad (7.15)$$

therefore, we rewrite (7.13), (7.14) as

$$\begin{cases} d_{-1}(p) := \frac{\partial p}{\partial \xi}(-1) = \frac{1}{\tau} (p(-1 + \tau) - p(-1)), \\ d_{-1}^2(p) := \frac{\partial^2 p}{\partial \xi^2}(-1) = \frac{1}{\tau^2} (p(-1 + \tau) - 2p(-1) + p(-1 - \tau)) \\ \quad = \frac{1}{\tau^2} (p(-1 + \tau) - 2p(-1) + p(1)), \quad (p(-1 - \tau) = p(1)), \\ \tilde{d}_{-1}^2(p) := \frac{1}{\tau^2} (p(-1 + \tau) + p(1)), \quad (p(-1 - \tau) = p(1)), \end{cases} \quad (7.16)$$

$$\begin{cases} d_1(p) := \frac{\partial p}{\partial \xi}(1) = \frac{1}{\tau}(p(1) - p(1 - \tau)), \\ d_1^2(p) := \frac{\partial^2 p}{\partial \xi^2}(1) = \frac{1}{\tau^2}(p(1 + \tau) - 2p(1) + p(1 - \tau)) \\ \quad = \frac{1}{\tau^2}(p(-1) - 2p(1) + p(1 - \tau)), \quad (p(1 + \tau) = p(-1)), \\ \tilde{d}_1^2(p) := \frac{1}{\tau^2}(p(-1) + p(1 - \tau)). \end{cases} \quad (7.17)$$

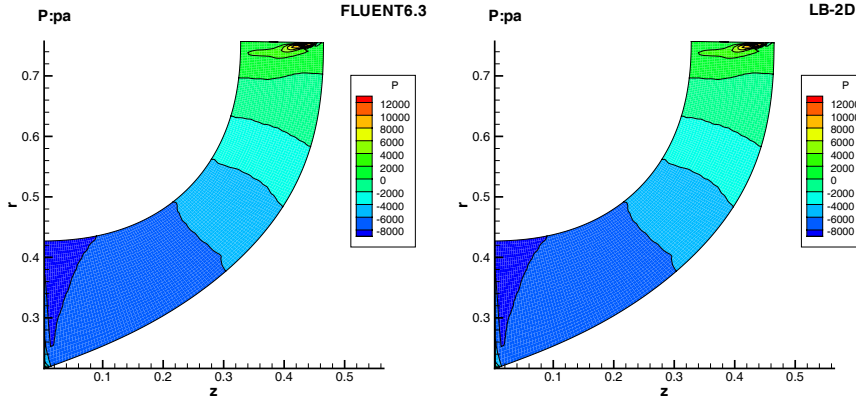
Summing up and introducing $p_- = p(-1)$, from (7.5) the equation of the pressure on the $\xi = -1$ surface is give by

$$\begin{cases} -\frac{1}{r} \frac{\partial}{\partial x^\alpha} (r \frac{\partial p_-}{\partial x^\alpha}) + \alpha_\tau p_- = f_-(\tau), \\ f_-(\tau) = -2r\Theta_\lambda d_{-1}(\frac{\partial p}{\partial x^\lambda}) - (\Theta_2 + r\tilde{\Delta}\Theta)d_{-1}(p) + \frac{a}{r\varepsilon} \tilde{d}_{-1}^2(p) \\ \quad + d_{-1}^2(w^3) - 2|\omega|^2, \end{cases} \quad (7.18)$$

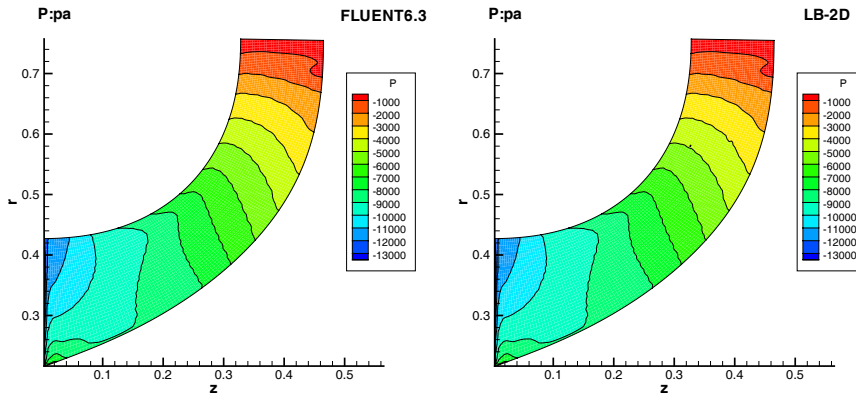
By a similar manner we can obtain the equation of pressure on the surface $\xi = 1$

$$\begin{cases} -\frac{1}{r} \frac{\partial}{\partial x^\alpha} (r \frac{\partial p_+}{\partial x^\alpha}) + \alpha_\tau p_+ = f_+(\tau), \\ f_+(\tau) = -2r\Theta_\lambda d_1(\frac{\partial p}{\partial x^\lambda}) - (\Theta_2 + r\tilde{\Delta}\Theta)d_1(p) + \frac{a}{\tau^2 r \varepsilon} \tilde{d}_1^2(p) \\ \quad + d_1^2(w^3) - 2|\omega|^2 \end{cases} \quad (7.19)$$

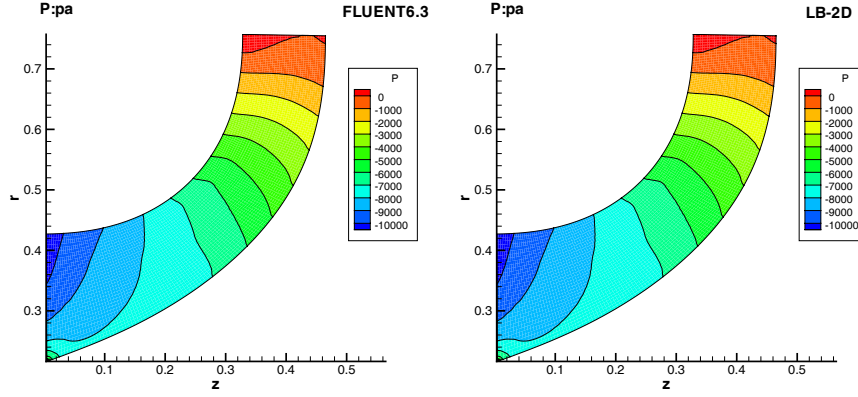
Next, in order to inspect the reliability of the method, we make some numerical simulations of pressure field by using the pressure correction equations on the blade surface (7.18), (7.19), and boundary conditions in (7.9). The low speed large-scale centrifugal impeller of The NASA is used as the example([22]), and some comparison with FLUENT's conclusions is diagramed as below,



(a) Pressure distribution on the pressure surface



(b) Pressure distribution on the suction surface



(c) Pressure distribution on the median surface

Fig5. Numerical comparison of the Fluent's conclusions and the Pressure Correction method
[Numerical results was completed by Chen Hao, Energy and Power Engineering College of Xi'an
Jiaotong University]

Where the LB-2D on the RHS are the results from the Pressure Correction method.

Fluid power, Fluent evaluates to 13973 watt, our method is 13975 watt.

8 Steam Layer in Domain decomposition and the Bi-parallel Algorithm

Next we consider the decomposition of the flow passage,

$$\Omega = D \times \{-1, 1\} = \sum_k \{D \times [\xi_k, \xi_{k+1}]\} = \sum_k \Omega_k$$

where $-1 = \xi_0 < \xi_1 < \dots < \xi_m = 1$. In the subsequent Ω_k is called “stream layer”.

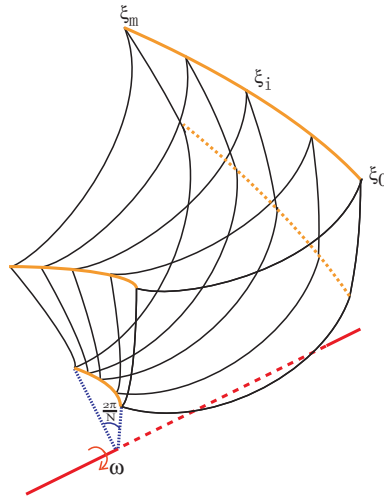


Fig 3: decomposition of the flow passage and angular expansion

The new method is to solve the 2D problem (6.10) with respect to the velocity and the pressure (w, p) and the pressure correction equation (7.9) on the interface \mathfrak{I}_k of two stream layer Ω_k and

$\Omega_k + 1$, which is a 2D manifold. The general parallel algorithm can be used to solve these 2D-problems (6.10) and (7.9).

Bi-Parallel Algorithm: The Bi-Parallel Algorithm means that we adopt the parallel algorithm to solve the problems (6.10) and (7.9) along the two directions, i.e., on the 2D manifold \mathfrak{S}_k and along the direction ξ . On the specified 2D manifold \mathfrak{S}_k , the general domain decomposition method or data parallel algorithms can be used to implement the parallel algorithm. On the other hand, the problems on the 2D manifold \mathfrak{S}_k corresponding to different discrete parameter $\xi_k, k = 1, \dots, m$, can be solved at the same time, which forms another parallel.

As new method is applied to solve 3D-viscous flow in turbo-machinery, all interfaces \mathfrak{S}_{ξ_k} have the same geometry properties, i.e., the same $a_{\alpha\beta}, b_{\alpha\beta}, \dots$. On the other hand, when this methods is applied to other 3D-flow, for example, circulation flow through the aircraft, geophysical flow around the earth, in this case the interface surface \mathfrak{S}_{ξ_k} have different geometry properties. Suppose the next interface $\mathfrak{S}_{\xi_{k+1}}$ is generated by a displacement $\boldsymbol{\eta}$ of the previous interface \mathfrak{S}_{ξ_k} , then the new fundamental forms $(a_{\alpha\beta}(\boldsymbol{\eta}), b_{\alpha\beta}(\boldsymbol{\eta}))$ can be computed by the following formulas,

Theorem 8.1. *Assume that \mathfrak{S} is a smooth surface in \mathbb{R}^3 , $a_{\alpha\beta}, b_{\alpha\beta}$ are metric and curvature tensors respectively. Given a smooth displacement field $\boldsymbol{\eta} = \eta^\alpha \mathbf{e}_\alpha + \eta^3 \mathbf{n}$ of \mathfrak{S} , we get the new surface $\mathfrak{S}(\boldsymbol{\eta})$, and use symbols $a_{\alpha\beta}(\boldsymbol{\eta}), b_{\alpha\beta}(\boldsymbol{\eta})$ to denote the metric tensor and the curvature tensors of the surface $\mathfrak{S}(\boldsymbol{\eta})$. Then a simple calculation shows that they can be expressed as (see [3,4]),*

$$\begin{cases} a_{\alpha\beta}(\boldsymbol{\eta}) &= a_{\alpha\beta} + 2 \overset{0}{E}_{\alpha\beta}(\boldsymbol{\eta}), \\ b_{\alpha\beta}(\boldsymbol{\eta}) &= b_{\alpha\beta} + \rho_{\alpha\beta}(\boldsymbol{\eta}) + Q_{\alpha\beta}^2(\boldsymbol{\eta}), \\ Q_{\alpha\beta}^2(\boldsymbol{\eta}) &= (b_{\alpha\beta} + \rho_{\alpha\beta}(\boldsymbol{\eta}))(q(\boldsymbol{\eta}) - 1) + q(\boldsymbol{\eta})[\phi_{\alpha\beta}(\boldsymbol{\eta})d(\boldsymbol{\eta}) + \phi_{\alpha\beta}^\sigma(\boldsymbol{\eta})m_\sigma(\boldsymbol{\eta}) \\ &\quad - (\rho_{\alpha\beta}^\sigma(\boldsymbol{\eta}) + \Gamma_{\alpha\beta}^\lambda \overset{0}{\nabla}_\lambda \eta^\sigma) \overset{0}{\nabla}_\sigma \eta^3], \end{cases} \quad (8.1)$$

where

$$\begin{cases} \overset{0}{E}_{\alpha\beta}(\boldsymbol{\eta}) &= \gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{1}{2}[a_{\lambda\sigma} \overset{0}{\nabla}_\alpha \eta^\lambda \overset{0}{\nabla}_\beta \eta^\sigma + \overset{0}{\nabla}_\alpha \eta^3 \overset{0}{\nabla}_\beta \eta^3], \\ \rho_{\alpha\beta}(\boldsymbol{\eta}) &= \overset{*}{\nabla}_\alpha \overset{0}{\nabla}_\beta \eta^3 + b_{\alpha\sigma} \overset{0}{\nabla}_\beta \eta^\sigma, \quad \gamma_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}(a_{\beta\lambda} \overset{0}{\nabla}_\alpha \eta^\lambda + a_{\alpha\lambda} \overset{0}{\nabla}_\beta \eta^\lambda), \\ \overset{0}{\nabla}_\beta \eta^\sigma &= \overset{*}{\nabla}_\beta \eta^\sigma - b_\beta^\sigma \eta^3, \quad \overset{*}{\nabla}_\beta \eta^\sigma = \partial_\beta \eta^\sigma - \Gamma_{\alpha\beta}^\sigma \eta^\alpha, \end{cases} \quad (8.2)$$

$$\begin{cases} \rho_{\alpha\beta}^\sigma(\boldsymbol{\eta}) &= \overset{*}{\nabla}_\alpha \overset{0}{\nabla}_\beta \eta^\sigma - b_\beta^\sigma \overset{0}{\nabla}_\alpha \eta^3, \\ \phi_{\alpha\beta}(\boldsymbol{\eta}) &= b_{\alpha\beta} + \rho_{\alpha\beta}(\boldsymbol{\eta}) + \Gamma_{\alpha\beta}^\lambda \overset{0}{\nabla}_\lambda \eta^3, \quad \phi_{\alpha\beta}^\sigma(\boldsymbol{\eta}) = \rho_{\alpha\beta}^\sigma(\boldsymbol{\eta}) + (\overset{0}{\nabla}_\lambda \eta^\sigma + \delta_\lambda^\sigma) \Gamma_{\alpha\beta}^\lambda, \end{cases} \quad (8.3)$$

$$\begin{cases} d_\sigma(\boldsymbol{\eta}) &= m_\sigma(\boldsymbol{\eta}) - \overset{0}{\nabla}_\sigma \eta^3, \quad d_0(\boldsymbol{\eta}) = 1 + d(\boldsymbol{\eta}), \\ d(\boldsymbol{\eta}) &= \gamma_0(\boldsymbol{\eta}) + \det(\overset{0}{\nabla}_\alpha \eta^\beta), \\ m_\sigma(\boldsymbol{\eta}) &= \varepsilon^{\nu\mu} \varepsilon_{\sigma\lambda} \overset{0}{\nabla}_\nu \eta^\lambda \overset{0}{\nabla}_\mu \eta^3, \\ m_1(\boldsymbol{\eta}) &= 2 \begin{vmatrix} \overset{0}{\nabla}_1 \eta^2 & \overset{0}{\nabla}_2 \eta^2 \\ \overset{0}{\nabla}_1 \eta^3 & \overset{0}{\nabla}_2 \eta^3 \end{vmatrix}, \quad m_2(\boldsymbol{\eta}) = -2 \begin{vmatrix} \overset{0}{\nabla}_1 \eta^1 & \overset{0}{\nabla}_2 \eta^1 \\ \overset{0}{\nabla}_1 \eta^3 & \overset{0}{\nabla}_2 \eta^3 \end{vmatrix}, \end{cases} \quad (8.4)$$

where $Q_{\alpha\beta}^2(\boldsymbol{\eta})$ is a remainder term which order is higher than 1.

Proof: The proof is omitted. \square

9 Existence of Solution of the 2D-3C Variational Problem

In this section we study the 2D-3C variational problem (6.10) on the manifold \mathfrak{S}_ξ . First, let $\partial D = \gamma_s \cup \gamma_0$ and introduce the Sobolev space $V(D)$ defined by

$$V(D) = \{\mathbf{w} |, \mathbf{w} = (w^\alpha, w^3) \in H^1(D) \times H^1(D) \times H^1(D), \mathbf{w}|_{\gamma_s} = 0\}, \quad (9.1)$$

equipped with the usual Sobolev norms

$$\begin{cases} \|\mathbf{w}\|_{1,D}^2 = \sum_{\alpha} \sum_j \|\partial_{\alpha} w^j\|_{0,D}^2, & \|\mathbf{w}\|_{0,D}^2 = \sum_i \|w^i\|_{0,D}^2 = \sum_i \int_D |w^i|^2 dx, \\ \|\mathbf{w}\|_{1,D}^2 = \|w\|_{1,D}^2 + \|\mathbf{w}\|_{0,D}^2. \end{cases} \quad (9.2)$$

It is clear that the variational problem (6.10) is a saddle point problem. In order to regularize it we introduce the artificial viscosity η such that

$$\begin{cases} \text{Find } \mathbf{w} \in V(D), p \in L^2(D), \text{ such that} \\ (\alpha_{\tau} \mathbf{w}, \mathbf{v}) + a_0(\mathbf{w}, \mathbf{v}) + (\mathbf{L}(\mathbf{w}, \Theta), \mathbf{v}) + (\mathbf{C}(\mathbf{w}, \omega), \mathbf{v}) + b(\mathbf{w}, \mathbf{w}, \mathbf{v}) - (p, \widetilde{\text{div}} \mathbf{v}) \\ = \langle \mathbf{G}_{\tau}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in V(D), \\ \eta(p, q) + (\text{div}_2 \mathbf{w} - d_{\tau}, q) = 0, \quad \forall q \in L^2(D), \end{cases} \quad (9.3)$$

where

$$\langle \mathbf{G}_{\tau}, \mathbf{v} \rangle = (\mathbf{f}_{\tau}, \mathbf{v}) + \langle \mathbf{h}, \mathbf{v} \rangle|_{\gamma_1}, \quad (9.4)$$

Obviously, problem (9.3) is equivalent to

$$\begin{cases} \text{Find } \mathbf{w} \in V(D), \text{ such that} \\ A_0(\mathbf{w}, \mathbf{v}) + \eta^{-1}(\text{div}_2 \mathbf{w}, \widetilde{\text{div}} \mathbf{v}) + b(\mathbf{w}, \mathbf{w}, \mathbf{v}) = \langle \mathbf{G}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in V(D), \\ p = \eta^{-1}[-\text{div}_2 \mathbf{w} + d_{\tau}(\mathbf{w})], \end{cases} \quad (9.5)$$

where

$$\begin{cases} A_0(\mathbf{w}, \mathbf{v}) = (\alpha_{\tau} \mathbf{w}, \mathbf{v}) + a_0(\mathbf{w}, \mathbf{v}) + (\mathbf{L}(\mathbf{w}, \Theta), \mathbf{v}) + (\mathbf{C}(\mathbf{w}, \omega), \mathbf{v}), \\ \langle \mathbf{G}, \mathbf{v} \rangle = (\mathbf{f}_{\tau}, \mathbf{v}) + \langle \mathbf{h}_{in}, \mathbf{v} \rangle|_{\Gamma_{in}} + (\eta^{-1} d_{\tau}, \widetilde{\text{div}} \mathbf{v}) - \langle \eta^{-1} d_{\tau}, \mathbf{v} \rangle, \end{cases} \quad (9.6)$$

Our first objective is to show that the bilinear form $A_0(\cdot, \cdot)$ defined by (9.6) is $V(D)$ -elliptic.

Theorem 9.1. *Let D be a bounded domain in R^2 , the injective mapping $\mathfrak{R}(x)$ defined by (2.1) satisfies $\mathfrak{R} \in C^3(\bar{D})$, $|D^i \mathfrak{R}|_{\infty, D} \leq k_0$, and the two vectors $\mathbf{e}_{\alpha} = \partial_{\alpha} \mathfrak{R}$ are linearly independent at all points of \bar{D} . Let γ_0 be a $d\gamma$ -measurable subset of $\gamma = \partial D$ and $d\gamma_0 > 0$. Then there exist the constants $C(k_0), C_0(k_0)$ depend upon x such that the following equality and inequalities hold*

$$\begin{cases} (i) & a_0(\mathbf{w}, \mathbf{v}) = a_0(\mathbf{v}, \mathbf{w}), & \forall \mathbf{w}, \mathbf{v} \in V(D), \\ (ii) & |a_0(\mathbf{w}, \mathbf{v})| \leq \nu(1 + r_1^2 k_0^2) \|\mathbf{w}\|_{1,D} \|\mathbf{v}\|_{1,D}, & \forall \mathbf{w}, \mathbf{v} \in V(D), \\ (iii) & |a_0(\mathbf{w}, \mathbf{w})| \geq \nu \|\mathbf{w}\|_{1,D}^2, & \forall \mathbf{w}, \mathbf{v} \in V(D), \end{cases} \quad (9.7)$$

$$\begin{cases} (iv) & |A_0(\mathbf{w}, \mathbf{v})| \leq C(k_0) \|\mathbf{w}\|_{1,D} \|\mathbf{v}\|_{0,D}, & \forall \mathbf{w}, \mathbf{v} \in V(D), \\ (v) & A_0(\mathbf{w}, \mathbf{w}) \geq \left(\frac{2}{r_1^2 \varepsilon^2 \tau^2} - C_0(k_0) k_0 \right) \|\mathbf{w}\|_{0,D}^2 + (\nu - C_1(k_0) k_0) \|\mathbf{w}\|_{1,D}^2 \\ & \quad + \int_D \frac{1}{r} \partial_r (w^{\alpha} w^{\alpha}) dx, & \forall \mathbf{w} \in V(D), \end{cases} \quad (9.8)$$

Proof: Firstly, from (6.11), we have

$$\begin{aligned} a_0(\mathbf{w}, \mathbf{v}) &= \int_D [\nu g_{ij} \tilde{\nabla}_{\lambda} w^i \tilde{\nabla}_{\lambda} v^j] dx = \int_D [\nu a_{\alpha\beta} \tilde{\nabla}_{\lambda} w^{\alpha} \tilde{\nabla}_{\lambda} v^{\beta} \\ &\quad + (\varepsilon r \tilde{\nabla}_{\lambda} w^3)(r \Theta_{\beta} \tilde{\nabla}_{\lambda} v^{\beta}) + (\varepsilon r \tilde{\nabla}_{\lambda} v^3)(r \Theta_{\beta} \tilde{\nabla}_{\lambda} w^{\beta}) + (\varepsilon r \tilde{\nabla}_{\lambda} w^3)(r \varepsilon \tilde{\nabla}_{\lambda} v^3)] dx \\ &= \nu \int_D [\delta_{\alpha\beta} \tilde{\nabla}_{\lambda} w^{\alpha} \tilde{\nabla}_{\lambda} v^{\beta} + (r \Theta_{\alpha} \tilde{\nabla}_{\lambda} w^{\alpha})(r \Theta_{\beta} \tilde{\nabla}_{\lambda} v^{\beta}) \\ &\quad + (r \Theta_{\beta} \tilde{\nabla}_{\lambda} w^{\beta})(r \varepsilon \tilde{\nabla}_{\lambda} v^3) + (r \varepsilon \tilde{\nabla}_{\lambda} w^3)(r \Theta_{\beta} \tilde{\nabla}_{\lambda} v^{\beta}) + (r \varepsilon \tilde{\nabla}_{\lambda} w^3)(r \varepsilon \tilde{\nabla}_{\lambda} v^3)] dx, \end{aligned}$$

By combining we have

$$a_0(\mathbf{w}, \mathbf{v}) = \nu \int_D [\delta_{\alpha\beta} \tilde{\nabla}_{\lambda} w^{\alpha} \tilde{\nabla}_{\lambda} v^{\beta} + (r \Theta_{\alpha} \tilde{\nabla}_{\lambda} w^{\alpha} + r \varepsilon \tilde{\nabla}_{\lambda} w^3)(r \Theta_{\beta} \tilde{\nabla}_{\lambda} v^{\beta} + r \varepsilon \tilde{\nabla}_{\lambda} v^3)] dx \quad (9.9)$$

hence, we get

$$a_0(w, v) = a_0(v, w), \quad |a_0(w, v)| \leq \nu(1 + r_1^2 k_0^2) |w|_{1,D} |v|_{1,D}.$$

$$a_0(w, w) = \nu |w|_{1,D}^2 + \nu(r\Theta_\alpha \tilde{\nabla}_\lambda w^\alpha + r\varepsilon \tilde{\nabla}_\lambda w^3)(r\Theta_\alpha \tilde{\nabla}_\lambda w^\alpha + r\varepsilon \tilde{\nabla}_\lambda w^3) \geq \nu |w|_{1,D}^2,$$

Then (9.7) is proved. Next we consider (9.8). Obviously (iv) is valid, thus we just need to prove (v). Indeed from (3.26), (3.27) and (3.35) we assert

$$\left\{ \begin{array}{l} L_\sigma(\mathbf{w}, \Theta) = (r^{-1}\delta_{2\beta}\delta_{\alpha\sigma} + r^2\Theta_\sigma\Theta_{\alpha\beta} - r^2\Theta_\alpha\Theta_{\beta\sigma})\partial_\beta w^\alpha - (2r\varepsilon\Theta_\beta a_{2\sigma} + \varepsilon r^2\Theta_{\beta\sigma})\partial_\beta w^3 \\ \quad + (a_{\alpha\sigma}q_m^\alpha + r^2\varepsilon q_m^3)w^m, \\ L_3(\mathbf{w}, \Theta) = r^2\varepsilon\Theta_{\alpha\beta}\partial_\beta w^\alpha - 2\varepsilon^2 r^3\Theta_2\Theta_\beta\partial_\beta w^3 + (\varepsilon r^2\Theta_\alpha q_m^\alpha + r^2\varepsilon^2 q_m^3)w^m, \\ C_\sigma(\mathbf{w}, \omega) = 2r\omega(w^2\Theta_\sigma - \delta_{2\sigma}\Pi(\mathbf{w}, \Theta)), \quad C_3(\mathbf{w}, \omega) = 2r\varepsilon\omega w^2, \\ a_{\alpha\sigma}q_\beta^\alpha + r^2\varepsilon q_\beta^3 = r^{-2}\delta_{2\beta}\delta_{2\sigma} + \delta_{2\beta}\Theta_2\Theta_\sigma + 2a_{2\sigma}(\delta_{2\beta}|\tilde{\nabla}\Theta|^2 - a\Theta_2\Theta_\beta \\ \quad + r\Theta_\beta\tilde{\Delta}\Theta - r\Theta_\lambda\Theta_{\lambda\beta}) + aa_{22}\Theta_\sigma\Theta_\beta + 2r\Theta_\sigma\Theta_{2\beta} + r^2\Theta_\sigma\partial_\beta\tilde{\Delta}\Theta, \\ a_{\alpha\sigma}q_3^\alpha + r^2\varepsilon q_3^3 = ra_{2\sigma}\tilde{\Delta}\Theta + \varepsilon a\Theta_2(r\Theta_2 - 2a_{2\sigma}), \\ (\varepsilon r^2\Theta_\alpha q_\beta^\alpha + r^2\varepsilon^2 q_\beta^3) = \varepsilon(\Theta_2\delta_{2\beta} + \Theta_\beta) + 2r\varepsilon\Theta_{2\beta} + r^2\varepsilon\Theta_2(2\delta_{2\beta}|\tilde{\nabla}\Theta|^2 - a\Theta_2\Theta_\beta) \\ \quad + \varepsilon r^3\Theta_2\Theta_\beta\tilde{\Delta}\Theta - r^3\varepsilon\Theta_2\Theta_\lambda\Theta_{\lambda\beta}, \\ (\varepsilon r^2\Theta_\alpha q_3^\alpha + r^2\varepsilon^2 q_3^3) = r^2\varepsilon^2(r\tilde{\Delta}\Theta - a\Theta_2), \end{array} \right. \quad (9.10)$$

therefore,

$$\begin{aligned} L_j(\mathbf{w}, \Theta)w^j &= [(r^{-1}\delta_{2\beta}\delta_{\alpha\sigma} + r^2\Theta_\sigma\Theta_{\alpha\beta} - r^2\Theta_\alpha\Theta_{\beta\sigma})\partial_\beta w^\alpha - (2r\varepsilon\Theta_\beta a_{2\sigma} + \varepsilon r^2\Theta_{\beta\sigma})\partial_\beta w^3]w^\sigma \\ &\quad + [(r^{-2}\delta_{2\beta}\delta_{2\sigma} + \delta_{2\beta}\Theta_2\Theta_\sigma + 2a_{2\sigma}(\delta_{2\beta}|\tilde{\nabla}\Theta|^2 - a\Theta_2\Theta_\beta \\ &\quad + r\Theta_\beta\tilde{\Delta}\Theta - r\Theta_\lambda\Theta_{\lambda\beta}) + aa_{22}\Theta_\sigma\Theta_\beta + 2r\Theta_\sigma\Theta_{2\beta} + r^2\Theta_\sigma\partial_\beta\tilde{\Delta}\Theta)w^\beta \\ &\quad + (ra_{2\sigma}\tilde{\Delta}\Theta + \varepsilon a\Theta_2(r\Theta_\sigma - 2a_{2\sigma}))w^3]w^\sigma \\ &\quad + [r^2\varepsilon\Theta_{\alpha\beta}\partial_\beta w^\alpha - 2\varepsilon^2 r^3\Theta_2\Theta_\beta\partial_\beta w^3]w^3 \\ &\quad + [(\varepsilon(\Theta_2\delta_{2\beta} + \Theta_\beta) + 2r\varepsilon\Theta_{2\beta} + r^2\varepsilon\Theta_2(2\delta_{2\beta}|\tilde{\nabla}\Theta|^2 - a\Theta_2\Theta_\beta) \\ &\quad + \varepsilon r^3\Theta_2\Theta_\beta\tilde{\Delta}\Theta - r^3\varepsilon\Theta_2\Theta_\lambda\Theta_{\lambda\beta})w^\beta + (r^2\varepsilon^2(r\tilde{\Delta}\Theta - a\Theta_2))w^3]w^3 \end{aligned}$$

By simplifying we get

$$\begin{aligned} L_j(\mathbf{w}, \Theta)w^j &= r^{-1}\frac{\partial}{\partial r}(w^\alpha w^\alpha) + r^{-2}w^2w^2 + P_\alpha^\beta(w, \Theta)\partial_\beta w^\alpha + P_3^\beta(w, \Theta)\partial_\beta w^3 \\ &\quad + Q_{\alpha\beta}(\Theta)w^\beta w^\alpha + Q_{3\beta}(\Theta)w^\beta w^3 + Q_{33}(\Theta)w^3w^3 \end{aligned}$$

where

$$\left\{ \begin{array}{l} P_\alpha^\beta(\mathbf{w}, \Theta) = r^2\varepsilon\Theta_{\alpha\beta}w^3 + r^2(\Theta_\sigma\Theta_{\alpha\beta} - \Theta_\alpha\Theta_{\beta\sigma})w^\sigma, \\ P_3^\beta(\mathbf{w}, \Theta) = -\varepsilon[2r^3\varepsilon\Theta_2\Theta_\beta w^3 + 2r^2(\Theta_{\beta\sigma} + r\Theta\Theta_2\sigma\Theta_\beta)w^\sigma, \\ Q_{\alpha\beta}(\Theta) = \delta_{2\beta}(\Theta_2\Theta_\alpha + 2a_{2\alpha}|\tilde{\nabla}\Theta|^2) + a\Theta_\beta(a_{22}\Theta_\alpha - 2a_{2\alpha}\Theta_2) \\ \quad + (2r\varepsilon\delta_{2\lambda}\Theta_\alpha + 2a_{2\alpha}\Theta_\lambda)\Theta_{\lambda\beta} + r\Theta_\beta(2a_{2\alpha} + r\Theta_\alpha)\tilde{\Delta}\Theta, \\ Q_{3\beta}(\Theta) = \varepsilon[r(2a_{2\beta} - \delta_{2\beta})\tilde{\Delta}\Theta - r^3\Theta_2\Theta_\lambda\Theta_{\lambda\beta} + 3ra\Theta_2\Theta_\beta + \Theta_\beta \\ \quad - ((1+a)\delta_{2\beta} + aa_{2\beta})\Theta_2], \\ Q_{33}(\Theta) = \varepsilon^2 r^2(r\tilde{\Delta}\Theta - a\Theta_2), \end{array} \right. \quad (9.11)$$

From the assumptions of the Lemma, we clare that there exist two constants $C_i(k_0), i = 0, 1$ independent of \mathbf{w}, Θ such that

$$(L(\mathbf{w}, \Theta), \mathbf{w}) \geq \int_D \frac{1}{r} \partial_r(w^\alpha w^\alpha) dx - C_0(k_0)k_0 \|\mathbf{w}\|_{0,D} - C_1(k_0)k_0 |\mathbf{w}|_{1,D}^2$$

In addition

$$(C(w, \omega), \mathbf{w}) = 2 \int_D (\omega \times \mathbf{w}) \mathbf{w} dx = 0,$$

Then, from (9.6) and $\alpha_\tau = \frac{\nu a}{\gamma^2 \varepsilon^2 \tau^2} \geq \frac{\nu}{r_1^2 \varepsilon^2 \tau^2}$, we get

$$\begin{aligned} A_0(\mathbf{w}, \mathbf{w}) &= (\alpha_\tau \mathbf{w}, \mathbf{w}) + a_0(\mathbf{w}, \mathbf{w}) + (\mathbf{L}(\mathbf{w}, \Theta), \mathbf{w}) + (\mathbf{C}(\mathbf{w}, \omega), \mathbf{w}) \\ &\geq (\frac{\nu}{r_1^2 \varepsilon^2 \tau^2} - C_0(k_0)k_0) \|\mathbf{w}\|_{0,D}^2 + (\nu - C_1(k_0)k_0) |\mathbf{w}|_{1,D}^2 + \int_D \frac{1}{r} \partial_r (w^\alpha w^\alpha) dx, \end{aligned}$$

the proof is ended. \square

Lemma 9.1. *Under the assumptions in Lemma 9.1, the trilinear form $b_0(\cdot, \cdot, \cdot)$ is continuous, i.e., there exists a constant $M(\Theta, D)$ independent of $\mathbf{w}, \mathbf{u}, \mathbf{v}$, such that*

$$|b(\mathbf{w}, \mathbf{u}, \mathbf{v})| \leq M \|\mathbf{w}\|_{H^{\frac{5}{6}}(D)} \|\mathbf{v}\|_{H^{\frac{5}{6}}(D)} \|\mathbf{u}\|_{1,D}, \forall \mathbf{w}, \mathbf{u}, \mathbf{v} \in V(D) \quad (9.12)$$

Proof: Thanks to the Hölder inequality

$$\int_D |w^\lambda \tilde{\nabla}_\lambda u^\alpha v^\beta| \sqrt{a} dx \leq \|\mathbf{w}\|_{L^4(D)} \|v^\beta\|_{L^4(D)} \|\tilde{\nabla} u^\alpha\|_{0,D}$$

and the Sobolev embedding theorems

$$\|\mathbf{u}\|_{L^4(D)} \leq C \|\mathbf{u}\|_{H^{\frac{5}{6}}(D)}, \quad \|\mathbf{u}\|_{L^3(\gamma_1)} \leq C \|\mathbf{u}\|_{H^{\frac{5}{6}}(D)}. \quad (9.13)$$

Further, by using the Cauchy's inequality, we derived the conclusion. The proof is completed. \square

Remark 9.1. *It is clear that we have*

$$\begin{aligned} a_{\alpha\beta} w^\lambda \overset{*}{\nabla}_\lambda w^\alpha w^\beta &= \overset{*}{\nabla}_\lambda (a_{\alpha\beta} w^\lambda u^\alpha v^\beta) - a_{\alpha\beta} w^\alpha \overset{*}{\nabla}_\lambda (w^\lambda w^\beta) \\ &= \overset{*}{div} (|\mathbf{w}| \mathbf{w}) - |\mathbf{w}| \overset{*}{div} \mathbf{w} - a_{\alpha\beta} w^\beta w^\lambda \overset{*}{\nabla}_\lambda w^\alpha, \end{aligned}$$

Hence

$$a_{\alpha\beta} w^\lambda \overset{*}{\nabla}_\lambda w^\alpha w^\beta = \frac{1}{2} \overset{*}{div} (|\mathbf{w}| \mathbf{w}) - \frac{1}{2} |\mathbf{w}| \overset{*}{div} \mathbf{w}.$$

Considering (6.6) and the boundary conditions of element in $V(D)$, we claim

$$|b_0(\mathbf{w}_0, \mathbf{w}_0, \mathbf{w}_0)| \leq C (\|\mathbf{w}_0\|_{L^4(D)}^2 \|\overset{*}{div} \mathbf{w}\|_{0,D}^2 + \|\mathbf{w}_0\|_{L^3(\gamma_1)}^3), \quad (9.14)$$

where the Gauss theorem is used.

Theorem 9.2. *Under the assumptions in Lemma 9.1, for the given $(G, d_0^3) \in V^*(D) \times H^{-1}(D)$, if \mathbf{F} satisfies the following condition,*

$$\|\mathbf{F}\|_* \leq \frac{\nu^2 \lambda^2}{MC^2}, \quad \text{with } \langle \mathbf{F}, \mathbf{v} \rangle = \langle \mathbf{G}, \mathbf{v} \rangle - \eta^{-1} (d_0^3, \overset{*}{div} \mathbf{v}), \quad (9.15)$$

then there exists one solution w_* of the variational problem (6.10) which satisfies

$$\|\mathbf{w}_*\|_{1,D} \leq \rho := \frac{\nu \lambda}{MC} - \sqrt{\left(\frac{\nu \lambda}{MC}\right)^2 - \frac{\|\mathbf{F}\|_*}{M}}. \quad (9.16)$$

Furthermore, if

$$\|\mathbf{F}\|_* < \frac{\nu^2 \lambda^2}{MC^2}, \quad (9.17)$$

then problem (6.10) has a unique solution in $V(D)$.

Proof: We begin with constructing a sequence of approximate solutions by Galerkin's method. Since the space $V(D)$ is separable, there exists a sequence $(\varphi_m, m \geq 1)$ in $V(D)$ such that: 1). for all $m \geq 1$, the elements $\varphi_1, \dots, \varphi_m$ are linearly independent; 2). the finite linear combinations $\sum_i c_i \varphi_i$ are dense in $V(D)$. Such a sequence $(\varphi_m, m \geq 1)$ is called a basis of the separable space $V(D)$.

Next we use V_m to denote the subspace of $V(D)$ spanned by finite sequence $\varphi_1, \dots, \varphi_m$. Then, we can construce the approximating problem,

$$\begin{cases} \text{Find } \mathbf{w}_m \in V_m \text{ such that} \\ A_0(\mathbf{w}_m, \mathbf{v}) + b_0(\mathbf{w}_m, \mathbf{w}_m, \mathbf{v}) = \langle \mathbf{F}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in V_m. \end{cases} \quad (9.18)$$

If we set

$$\mathbf{w}_m = \sum_{i=1}^m c_i \varphi_i,$$

then we find that problem (9.18) amounts to solve a system of m nonlinear equations with m unknowns c_i . For each m problem (9.18) has at least one solution. Indeed, we can introduce the mapping $\mathcal{M}_m : V_m \rightarrow V_m$,

$$(\mathcal{M}_m(\mathbf{u}), \varphi_i) = A_0(\mathbf{u}, \varphi_i) + b_0(\mathbf{u}, \mathbf{u}, \varphi_i) - \langle \mathbf{F}, \varphi_i \rangle, \quad 1 \leq i \leq m,$$

where (\cdot, \cdot) is the scalar product in V . Hence, $\mathbf{w}_m \in V_m$ is a solution of problem (9.18) if only if $\mathcal{M}_m(\mathbf{w}_m) = 0$. Since

$$(\mathcal{M}_m(\mathbf{u}), \mathbf{u}) = A_0(\mathbf{u}, \mathbf{u}) + b_0(\mathbf{u}, \mathbf{u}, \mathbf{u}) - \langle \mathbf{F}, \mathbf{u} \rangle, \quad \forall \mathbf{u} \in V_m,$$

it follows that

$$(\mathcal{M}_m(\mathbf{u}), \mathbf{u}) \geq \left(\frac{2\nu\lambda}{C} \|\mathbf{u}\|_{1,D} - M \|\mathbf{u}\|_{1,D}^2 - \|\mathbf{F}\|_* \right) \|\mathbf{u}\|_{1,D} \quad (9.19)$$

Hence, if choosing

$$\rho = \frac{\nu\lambda}{MC} - \sqrt{\left(\frac{\nu\lambda}{MC} \right)^2 - \frac{\|\mathbf{F}\|_*}{M}},$$

then we get for all $\mathbf{u} \in V_m$ with $\|\mathbf{u}\|_{1,D} = \rho$,

$$(\mathcal{M}_m(\mathbf{u}), \mathbf{u}) \geq 0.$$

Moreover, \mathcal{M}_m is continuous in V_m , and the space V_m is finite dimensional, we can apply Corollary 1.1 in [17], there exists at least one solution $\mathbf{w}_m \in V_m$ of problem (9.18).

Furthermore, we have for any solution \mathbf{w}_m to (9.18)

$$0 = (\mathcal{M}_m(\mathbf{w}_m), \mathbf{w}_m) \geq \left(\frac{2\nu\lambda}{C} \|\mathbf{w}_m\|_{1,D} - M \|\mathbf{w}_m\|_{1,D}^2 - \|\mathbf{F}\|_* \right) \|\mathbf{w}_m\|_{1,D},$$

therefore,

$$\frac{2\nu\lambda}{C} \|\mathbf{w}_m\|_{1,D} - M \|\mathbf{w}_m\|_{1,D}^2 - \|\mathbf{F}\|_* \leq 0.$$

It follows that, when denoting by $y = \|\mathbf{w}_m\|_{1,D}$,

$$\left(y - \frac{\nu\lambda}{MC} \right)^2 \geq \left(\frac{\nu\lambda}{MC} \right)^2 - \frac{\|\mathbf{F}\|_*}{M} \Rightarrow y - \frac{\nu\lambda}{MC} \leq -\sqrt{\left(\frac{\nu\lambda}{MC} \right)^2 - \frac{\|\mathbf{F}\|_*}{M}},$$

i.e.,

$$\|\mathbf{w}_m\|_{1,D} \leq \frac{\nu\lambda}{MC} - \sqrt{\left(\frac{\nu\lambda}{MC} \right)^2 - \frac{\|\mathbf{F}\|_*}{M}}. \quad (9.20)$$

This shows that the sequence (\mathbf{w}_m) is uniformly bounded in V . Therefore, we can extract a subsequence, still denoted by \mathbf{w}_m , such that

$$\mathbf{w}_m \rightharpoonup (\text{weak}) \mathbf{w}_* \text{ in } V(D) \text{ as } m \rightarrow +\infty.$$

Then, the compactness of the embedding of $V(D)$ into $L^2(D)^3$ implies that

$$\mathbf{w}_m \rightarrow (\text{strong}) \mathbf{w}_* \text{ in } L^2(D)^3 \text{ as } m \rightarrow +\infty,$$

the remainder is to prove $b(\cdot, \cdot, \cdot)$ is weakly sequence continuous, i.e., $b_0(\mathbf{w}_m, \mathbf{w}_m, \mathbf{v}) \rightarrow b_0(\mathbf{w}_*, \mathbf{w}_*, \mathbf{v})$. To do this, we recall

$$\mathcal{V} = \{ \mathbf{u} \in C^\infty(D) \text{ satisfy the boundary conditon (3.36)} \}$$

is dense in $V(D)$ and

$$b_0(\mathbf{w}_m, \mathbf{w}_m, \mathbf{v}) = \int_{\gamma_1} a_{\alpha\beta} w_m^\alpha v^\beta a_{\lambda\sigma} w_m^\lambda n^\sigma dl - b_0(\mathbf{w}_m, \mathbf{v}, \mathbf{w}_m).$$

For any $\mathbf{v} \in \mathcal{V}$, then $\mathbf{v} \in L^\infty(D) L^\infty(\gamma_1)$, $\partial_{x^\alpha} v^\beta \in L^\infty(D)$, and the convergence relations $\lim_{m \rightarrow \infty} w_m^\lambda w_m^\sigma = w_*^\lambda w_*^\sigma$ are satisfied in $L^1(D)$ and $L^1(\gamma_1)$ respectively, therefore

$$\lim_{m \rightarrow \infty} b_0(\mathbf{w}_m, \mathbf{w}_m, \mathbf{v}) = \int_{\gamma_1} a_{\alpha\beta} w_*^\alpha v^\beta a_{\lambda\sigma} w_*^\lambda n^\sigma dl - b_0(\mathbf{w}_*, \mathbf{v}, \mathbf{w}_*) = b_0(\mathbf{w}_*, \mathbf{w}_*, \mathbf{v}), \quad \forall \mathbf{v} \in \mathcal{V}.$$

Next for all $\mathbf{v} \in V(D)$, by virtue of the density of \mathcal{V} , and taking the limitation of both sides of (9.18) implies

$$A_0(\mathbf{w}_*, \mathbf{v}) + b_0(\mathbf{w}_*, \mathbf{w}_*, \mathbf{v}) = \langle \mathbf{F}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in V(D), \quad (9.21)$$

that means \mathbf{w}_* is a solution of problem (6.10).

In order to prove (9.16), by a similar manner we get

$$|\|\mathbf{w}_*\|_{1,D} - \frac{\nu\lambda}{MC}| \geq \sqrt{(\frac{\nu\lambda}{MC})^2 - \frac{\|\mathbf{F}\|_*}{M}},$$

$$\|\mathbf{w}_*\|_{1,D} \geq \frac{\nu\lambda}{MC} + \sqrt{(\frac{\nu\lambda}{MC})^2 - \frac{\|\mathbf{F}\|_*}{M}}, \text{ or } \|\mathbf{w}_*\|_{1,D} \leq \frac{\nu\lambda}{MC} - \sqrt{(\frac{\nu\lambda}{MC})^2 - \frac{\|\mathbf{F}\|_*}{M}}.$$

Obviously, the first one of the above inequalities is contract to (9.20), thus only the second one is true, that is (9.16).

Next we prove the uniqueness. In fact, if there exist two solutions \mathbf{w}_* and $\tilde{\mathbf{w}}_*$ of (6.10). Let $\mathbf{e}_* = \mathbf{w}_* - \tilde{\mathbf{w}}_*$, then

$$A_0(\mathbf{e}_*, \mathbf{e}_*) + b_0(\mathbf{e}_*, \mathbf{w}_*, \mathbf{e}_*) + b_0(\tilde{\mathbf{w}}_*, \mathbf{e}_*, \mathbf{e}_*) = 0.$$

Owing to condition satisfied by w_* and \tilde{w}_* and (6.17), we get

$$0 \geq (\frac{2\nu\lambda}{C} - 2M(\frac{\nu\lambda}{MC} - \sqrt{(\frac{\nu\lambda}{MC})^2 - \frac{\|\mathbf{F}\|_*}{M}}) \|\mathbf{e}_*\|_{1,D}^2 = 2\sqrt{(\frac{\nu\lambda}{MC})^2 - \frac{\|\mathbf{F}\|_*}{M}}.$$

This yields $\|\mathbf{e}_*\|_{1,D} = 0$. Therefor, the solutions is unique. The proof is complete. \square

Theorem 9.3. Let (\mathbf{w}_0, p_0) and $(\mathbf{w}_\eta, p_\eta)$ be the solutions of (6.10) and (9.3), respectively. If \mathbf{F} and $H_* = \sup_D |H|$ satisfy the condition

$$C_0 - 2M\rho - 2\eta^{-1}H_*^2 \geq C_2 > 0, \quad (9.22)$$

then the following estimates are valid

$$\|\mathbf{w}_0 - \mathbf{w}_\eta\|_{1,D} + \|p_0 - p_\eta\|_{0,D} \leq \max(C_3, C_4)\eta, \quad (9.23)$$

where

$$C_3 = \frac{C + 2M\rho}{C_2\beta_0} \|p_0\|_{0,D}, \quad C_4 = \frac{(C + 2M\rho)^2}{C_2\beta_0^2} \|p_0\|_{0,D}. \quad (9.24)$$

and β_0 is the constant in the inf – sup condition.

Proof: From the assumption we have

$$\begin{cases} \text{Find } \mathbf{w}_0 \in V(D), p_0 \in L^2(D), \text{ such that} \\ a_0(\mathbf{w}_0, v) - (p_0, \operatorname{div}^* \mathbf{v}) + b_0(\mathbf{w}_0, \mathbf{w}_0, \mathbf{v}) + (l(\mathbf{w}_0), \mathbf{v}) = \langle \mathbf{G}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in V(D), \\ (\operatorname{div}^* \mathbf{w}_0 - 2Hw_0^3 + d_0^3, q) = 0, \quad \forall q \in L^2(D), \end{cases} \quad (9.25)$$

and

$$\begin{cases} \text{Find } \mathbf{w}_\eta \in V(D), p_\eta \in L^2(D), \text{ such that,} \\ a_0(\mathbf{w}_\eta, v) - (p_\eta, \operatorname{div}^* \mathbf{v}) + b_0(\mathbf{w}_\eta, \mathbf{w}_\eta, \mathbf{v}) + (l(\mathbf{w}_\eta), \mathbf{v}) = \langle \mathbf{G}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in V(D) \\ \eta(p_\eta, q) + (\operatorname{div}^* \mathbf{w}_\eta - 2Hw_\eta^3 + d_0^3, q) = 0, \forall q \in L^2(D). \end{cases} \quad (9.26)$$

Next, we denote $e_* = w_0 - w_\eta, s_* = p_0 - p_\eta$. subtracting (9.25) from (9.26) then yields

$$\begin{cases} a_0(e_*, v) + b_0(e_*, \mathbf{w}_0, \mathbf{v}) + b_0(\mathbf{w}_\eta, e_*, \mathbf{v}) - (s_*, \operatorname{div}^* \mathbf{v}) = 0, \forall \mathbf{v} \in V(D), \\ (\operatorname{div}^* e_*, q) + (-2He_*^3, q) + \eta(s_*, q) - \eta(p_0, q) = 0, \forall q \in L^2(D). \end{cases} \quad (9.27)$$

choosing $\mathbf{v} = e_*, q = s_*$ in (9.27) and summing the above two equations, we obtain

$$a_0(e_*, e_*) + b_0(e_*, \mathbf{w}_0, e_*) + b_0(\mathbf{w}_\eta, e_*, e_*) + \eta(s_*, s_*) - \eta(p_0, s_*) - (2He_*^3, s_*) = 0. \quad (9.28)$$

Noting that (9.7), (9.8), and (9.12), we have

$$(C_0 - 2M\rho)\|e_*\|_{1,D}^2 + \eta\|s_*\|_{0,D}^2 \leq (\eta\|p_0\|_{0,D} + 2\sup_D |H| \|e_*\|_{0,D})\|s_*\|_{0,D}. \quad (9.29)$$

Furthermore by using Young's inequality we get

$$2\sup_D |H| \|e_*\|_{0,D} \|s_*\|_{0,D} \leq \frac{1}{2}\eta\|s_*\|_{0,D}^2 + 2\eta^{-1}H_*^2\|e_*\|_{1,D}^2, \quad (9.30)$$

therefore from (9.29), (9.30)

$$(C_0 - 2M\rho - 2\eta^{-1}H_*^2)\|e_*\|_{1,D}^2 + \frac{1}{2}\eta\|s_*\|_{0,D}^2 \leq \eta\|p_0\|_{0,D}\|s_*\|_{0,D},$$

Noting that the condition (9.22) are satisfied, hence

$$\|e_*\|_{1,D}^2 \leq \eta \frac{\|p_0\|_{0,D}}{C_2} \|s_*\|_{0,D}. \quad (9.31)$$

On the other hand, the inf – sup condition means that

$$\begin{aligned} \beta_0\|s_*\|_{0,D} &\leq \sup_{\mathbf{v} \in V(D)} \frac{|(s_*, \operatorname{div}^* \mathbf{v})|}{\|\mathbf{v}\|_{1,D}} \\ &\leq \sup_{\mathbf{v} \in V(D)} (\|\mathbf{v}\|_{1,D}^{-1} [a_0(e_*, \mathbf{v}) + b_0(e_*, \mathbf{w}_0, \mathbf{v}) + b_0(\mathbf{w}_\eta, e_*, \mathbf{v})]) \\ &\leq (C + 2M\rho)\|e_*\|_{1,D}. \end{aligned} \quad (9.32)$$

Finally from (9.31), (9.32) we have

$$\|\mathbf{e}_*\|_{1,D} \leq C_3 \eta^2, \quad \|s_*\|_{0,D} \leq C_4 \eta,$$

where

$$C_3 = \frac{C + 2M\rho}{C_2\beta_0} \|p_0\|_{0,D}, \quad C_4 = \frac{(C + 2M\rho)^2}{C_2\beta_0^2} \|p_0\|_{0,D}.$$

Thus the theorem is proved \square

10 Finite Element Approximation Based on Approximate Inertial Manifold

In this section, we focus on the variational problem for the 2D-3C problem (6.10), called 2D-3C variational problem,

$$\begin{cases} \text{Find } \mathbf{w}_0 \in V(D), \text{ such that} \\ A_0(\mathbf{w}_0, \mathbf{v}) + b_0(\mathbf{w}_0, \mathbf{w}_0, \mathbf{v}) = \langle \mathbf{G}_\eta, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in V(D), \end{cases} \quad (10.1)$$

where

$$\begin{cases} A_0(\mathbf{w}_0, \mathbf{v}) &= a_0(\mathbf{w}_0, \mathbf{v}) + \eta^{-1}(\operatorname{div}^* \mathbf{w}_0, \operatorname{div}^* \mathbf{v}) - \eta^{-1}(2Hw_0^3, \operatorname{div}^* \mathbf{v}) + (l_0(\mathbf{w}_0), \mathbf{v}) \\ &= a_0(w_0, v) + \eta^{-1}(\operatorname{div}^* w_0, \operatorname{div}^* v) + (\nu - \eta^{-1})(2Hw_0^3, \operatorname{div}^* v) \\ &\quad + (a_{\alpha\beta}(C_\lambda^\alpha w_0^\lambda + C_3^\alpha w_0^3), v^\beta) + (C_\beta^3 w_0^\beta, v^3), \\ a_0(\mathbf{w}_0, \mathbf{v}) &= 2\nu(a^{\alpha\lambda} a^{\beta\sigma} \gamma_{\lambda\sigma}(\mathbf{w}_0), \gamma_{\alpha\beta}(\mathbf{v})) + \nu(a^{\alpha\beta} \nabla_\alpha^* w_0^3, \nabla_\beta^* v^3) \\ &\quad + \nu h^{-2}[(a_{\alpha\beta} w_0^\alpha, v^\beta) + (w_0^3, v^3)], \\ b_0(\mathbf{w}_0, \mathbf{w}_0, \mathbf{v}) &= (a_{\alpha\beta} w_0^\lambda \nabla_\lambda^* w_0^\alpha - 2b_{\alpha\beta} w_0^\alpha w_0^3, v^\beta) + (w^\beta \nabla_\beta^* w_0^3 + b_{\alpha\beta} w_0^\alpha w_0^\beta, v^3), \\ (l(\mathbf{w}_0), \mathbf{v}) &= (a_{\alpha\beta} l^\alpha(w_0), v^\beta) + (l^3(w_0), v^3) \\ &= (2\nu H \nabla_\beta^* w_0^3, v^\beta) + (a_{\alpha\beta}(C_\lambda^\alpha w_0^\lambda + C_3^\alpha w_0^3), v^\beta) + (C_\beta^3 w_0^\beta, v^3), \\ \langle \mathbf{G}_\eta, \mathbf{v} \rangle &= \langle \mathbf{F}_h, \mathbf{v} \rangle + \int_{\gamma_0} [\sigma_{n\alpha} v^\alpha + \sigma_{n3} v^3] d\gamma + (a_{\alpha\beta} d_0^\alpha m_0^3, v^\beta) - \eta^{-1}(d_0^3, \operatorname{div}^* v) \end{cases} \quad (10.2)$$

We now consider the finite element approximation of the 2D-3C variational problem (10.1). Assume that V_h and M_h are finite element subspaces of $V(D)$ and $L^2(D)$ respectively. Introduce the product space $Y_h = V_h \times M_h$, obviously which is a subspace of $Y = V(D) \times L^2(D)$.

Then the standard Galerkin finite element approximation of (10.1) is defined by

$$\begin{cases} \text{Find } \mathbf{w}_h \in V_h, \text{ such that} \\ A_0(\mathbf{w}_h, \mathbf{v}) + b_0(\mathbf{w}_h, \mathbf{w}_h, \mathbf{v}) = \langle \mathbf{G}_h, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V_h \end{cases} \quad (10.3)$$

As usual, we make the following standard assumptions on the finite element subspace Y_h

(H1) Approximation property

$$\inf_{(\mathbf{v}_h, q_h) \in Y_h} \{h\|\mathbf{u} - \mathbf{v}_h\|_{1,D} + \|\mathbf{u} - \mathbf{v}_h\|_{0,D} + h\|p - q_h\|_{0,D}\} \leq C h^{k+1} \{\|\mathbf{u}\|_{k+1,D} + \|p\|_{k,D}\}$$

for any $(\mathbf{u}, p) \in Y \cap (H^{k+1}(\Omega)^d \times H^k(\Omega))$, $1 \leq k \leq l$.

(H2) Interpolation property

$$\|\mathbf{v} - I_h \mathbf{v}\|_{1,D} + \|q - J_h q\|_{0,D} \leq C h^k (\|\mathbf{v}\|_{k+1,D} + \|q\|_{k,D})$$

for any $(\mathbf{v}, q) \in Y \cap (H^{k+1}(\Omega)^d \times H^k(\Omega))$, $1 \leq k \leq l$, where I_h and J_h are some interpolation operators from $V(D)$ and $L^2(D)$ into X_h and M_h , respectively.

(H3) Inverse inequality

$$\|\mathbf{v}_h\|_{1,D} \leq C h^{-1} \|\mathbf{v}_h\|_{0,D} \quad \forall \mathbf{v}_h \in V_h.$$

(H4) (V_h, M_h) satisfies LBB-condition

$$\inf_{q \in M_h} \sup_{\mathbf{v} \in X_h} \frac{(q, \operatorname{div} \mathbf{v})}{\|q\|_{0,D} \|\mathbf{v}\|_{1,D}} \geq \beta > 0,$$

where β is a constant independent of h .

The following optimal error estimates of the Galerkin finite element approximation are well-known (cf.[17]),

Theorem 10.1. *Suppose $\mathbf{w}_0 \in V(D) \cap H^{k+1}(D)^3$ is a nonsingular solution of (10.1) and the finite element subspace V_h satisfies assumptions (H1) \sim (H4). Then there exists a solution \mathbf{w}_h satisfying (10.3) such that*

$$h \|\mathbf{w}_0 - \mathbf{w}_h\|_{1,D} + \|\mathbf{w}_0 - \mathbf{w}_h\|_{0,D} \leq C h^{k+1} \|\mathbf{w}_0\|_{k+1,D}. \quad (10.4)$$

Next we try to improve the error estimation. To do that we rewrite (10.1) in operator form. Let $\mathcal{F} : V(D) \longrightarrow V^*(D)$ denote the 2D-3C Navier-Stokes operator on the manifold \mathfrak{S} via

$$\langle \mathcal{F}(\mathbf{w}_0), \mathbf{v} \rangle := A_0(\mathbf{w}_0, \mathbf{v}) + b_0(\mathbf{w}_0, \mathbf{w}_0, \mathbf{v}) - \langle \mathbf{G}_\eta, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in V(D).$$

It is obvious that $\mathcal{F}(\mathbf{w}_0) = 0$ is equivalent to (10.1). The operator form of finite element approximation (10.3) is

$$\langle \mathcal{F}_h(\mathbf{w}_h), \mathbf{v} \rangle := A_0(\mathbf{w}_h, \mathbf{v}) + b_0(\mathbf{w}_h, \mathbf{w}_h, \mathbf{v}) - \langle \mathbf{G}_\eta, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in V_h(D).$$

Therefore, $\mathcal{F}_h(\mathbf{w}_h) = 0$ is equivalent to (10.3). Furthermore, it is easy to show that $\mathcal{F}(\mathbf{w}_0)$ and $\mathcal{F}_h(\mathbf{w}_h)$ are Fréchet differentiable and the Fréchet derivatives at \mathbf{w}_0 and \mathbf{w}_h along direction \mathbf{u} are given by, respectively

$$\begin{aligned} \mathcal{A}\mathbf{w}_0(\mathbf{u}, \mathbf{v}) &:= (D\mathcal{F}(\mathbf{w}_0)\mathbf{u}, \mathbf{v}) = A_0(\mathbf{u}, \mathbf{v}) + b_0(\mathbf{u}, \mathbf{w}_0, \mathbf{v}) + b_0(\mathbf{w}_0, \mathbf{u}, \mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V(D), \\ \mathcal{A}\mathbf{w}_h(\mathbf{u}, \mathbf{v}) &:= (D\mathcal{F}_h(\mathbf{w}_h)\mathbf{u}, \mathbf{v}) = A_0(\mathbf{u}, \mathbf{v}) + b_0(\mathbf{u}, \mathbf{w}_h, \mathbf{v}) + b_0(\mathbf{w}_h, \mathbf{u}, \mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V_h(D). \end{aligned}$$

It is well known that \mathbf{w}_0 is a nonsingular solution of (10.1) if and only if $D\mathcal{F}(\mathbf{w}_0)$ is an isomorphism on $V(D)$, furthermore, equivalent to $\mathcal{A}\mathbf{w}_0(\cdot, \cdot)$ satisfies the inf – sup condition (weak coerciveness), i.e.,

$$\inf_{\mathbf{u} \in V(D)} \sup_{\mathbf{v} \in V(D)} \frac{\mathcal{A}\mathbf{w}_0(\mathbf{u}, \mathbf{v})}{\|\mathbf{u}\|_{1,D} \|\mathbf{v}\|_{1,D}} \geq \alpha_0 > 0, \quad \inf_{\mathbf{v} \in V(D)} \sup_{\mathbf{u} \in V(D)} \frac{\mathcal{A}\mathbf{w}_0(\mathbf{u}, \mathbf{v})}{\|\mathbf{u}\|_{1,D} \|\mathbf{v}\|_{1,D}} \geq \alpha_0 > 0. \quad (10.5)$$

In this case, for any $\mathbf{f} \in V^*(D)$, the variational problem

$$\begin{cases} \text{Find } \mathbf{u} \in V(D) \text{ such that} \\ \mathcal{A}\mathbf{w}_0(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in V(D), \end{cases} \quad (10.6)$$

has a one and only one solution. Similarly, \mathbf{w}_h is a nonsingular solution of (10.3) if and only if $D\mathcal{F}_h(\mathbf{w}_h)$ is an isomorphism on $V_h(D)$, equivalent to $\mathcal{A}\mathbf{w}_h(\cdot, \cdot)$ satisfies the inf – sup condition (weak coerciveness)

$$\inf_{\mathbf{u} \in V_h(D)} \sup_{\mathbf{v} \in V_h(D)} \frac{\mathcal{A}\mathbf{w}_h(\mathbf{u}, \mathbf{v})}{\|\mathbf{u}\|_{1,D} \|\mathbf{v}\|_{1,D}} \geq \alpha_h > 0, \quad \inf_{\mathbf{v} \in V_h(D)} \sup_{\mathbf{u} \in V_h(D)} \frac{\mathcal{A}\mathbf{w}_h(\mathbf{u}, \mathbf{v})}{\|\mathbf{u}\|_{1,D} \|\mathbf{v}\|_{1,D}} \geq \alpha_h > 0, \quad (10.7)$$

In this case, the variational problem

$$\begin{cases} \text{Find } \mathbf{u}_h \in V_h(D) \text{ such that} \\ \mathcal{A}\mathbf{w}_h(\mathbf{u}_h, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in V_h(D), \end{cases} \quad (10.8)$$

has a unique solution w_h for any $\mathbf{f} \in V_h^*(D)$. Condition (10.5) is equivalent to

$$\|D\mathcal{F}(\mathbf{w}_0)\|_{\mathcal{L}(V,V)} \leq \alpha_0^{-1}. \quad (10.9)$$

The next theorem shows the uniqueness condition to insure the finite element solution \mathbf{w}_h of (10.3).

Theorem 10.2. *Assume that the assumptions (H1) \sim (H4) are valid, and \mathbf{w}_0 is a nonsingular solution of (10.1). If the finite element mesh h is small enough such that*

$$2MC\alpha_0^{-1}\|\mathbf{w}_0\|_{2,D}h < 1. \quad (10.10)$$

Then, solution \mathbf{w}_h of the finite element approximation problem (10.3) is nonsingular.

Proof: In fact, from the above explanation in this section, it is enough to prove that

$$\|D\mathcal{F}_h(\mathbf{w}_h)\|_{\mathcal{L}(V_h,V_h)} \leq \beta_0^{-1}. \quad (10.11)$$

Therefore, noting that

$$\begin{aligned} \varepsilon : &= \|D\mathcal{F}(\mathbf{w}_0) - D\mathcal{F}_h(\mathbf{w}_h)\|_{\mathcal{L}(V,V)} \\ &= \sup_{\mathbf{u}, \mathbf{v} \in V_h} \frac{((D\mathcal{F}(\mathbf{w}_0) - D\mathcal{F}_h(\mathbf{w}_h))\mathbf{u}, \mathbf{v})}{\|\mathbf{u}\|_{1,D}\|\mathbf{v}\|_{1,D}} \\ &= \sup_{\mathbf{u}, \mathbf{v} \in V} \frac{b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{u}, \mathbf{v}) + b_0(\mathbf{u}, \mathbf{w}_0 - \mathbf{w}_h, \mathbf{v})}{\|\mathbf{u}\|_{1,D}\|\mathbf{v}\|_{1,D}} \\ &\leq 2M\|\mathbf{w}_0 - \mathbf{w}_h\|_{1,D} \leq 2MC\|\mathbf{w}_0\|_{2,D}h. \end{aligned} \quad (10.12)$$

Set $B = \{D\mathcal{F}(\mathbf{w}_0)\}^{-1}\{D\mathcal{F}(\mathbf{w}_0) - D\mathcal{F}_h(\mathbf{w}_h)\}$. Then from (10.9) and (10.12), we have

$$\begin{aligned} D\mathcal{F}_h(\mathbf{w}_h) &= D\mathcal{F}(\mathbf{w}_0)(I - B), \\ \|B\|_{\mathcal{L}(V,V)} &\leq \alpha_0^{-1}2MC\|\mathbf{w}_0\|_{2,D}h, \quad \|(I - B)^{-1}\|_{\mathcal{L}(V,V)} \leq \frac{1}{1 - 2MC\alpha_0^{-1}\|\mathbf{w}_0\|_{2,D}h}, \\ \|D\mathcal{F}_h(\mathbf{w}_h)\|_{\mathcal{L}(V,V)} &\leq \frac{1}{\alpha_0} \frac{1}{1 - 2MC\alpha_0^{-1}\|\mathbf{w}_0\|_{2,D}h}. \end{aligned}$$

substituting (10.10) into the above inequality, then we get $D\mathcal{F}_h(\mathbf{w}_h)$ is an isomorphism on V_h , hence \mathbf{w}_h is a nonsingular solution of (10.3). \square

Theorem 10.2 shows if mesh size h is small enough, then we have

$$\inf_{\mathbf{u} \in V_h(D)} \sup_{\mathbf{v} \in V_h(D)} \frac{\mathcal{A}\mathbf{w}_h(\mathbf{u}, \mathbf{v})}{\|\mathbf{u}\|_{1,D}\|\mathbf{v}\|_{1,D}} \geq \frac{1}{2}\alpha_0 > 0, \quad \inf_{\mathbf{v} \in V_h(D)} \sup_{\mathbf{u} \in V_h(D)} \frac{\mathcal{A}\mathbf{w}_h(\mathbf{u}, \mathbf{v})}{\|\mathbf{u}\|_{1,D}\|\mathbf{v}\|_{1,D}} \geq \frac{1}{2}\alpha_0 > 0. \quad (10.13)$$

Next, assume \mathbf{w}_h is a nonsingular solution (10.3). We define a projection $P_h : V(D) \rightarrow V_h(D)$, $\forall \mathbf{w} \in V(D)$ through

$$\mathcal{A}\mathbf{w}_h(\mathbf{w} - P_h\mathbf{w}, \mathbf{v}) = 0, \forall \mathbf{v} \in V_h(D). \quad (10.14)$$

Since \mathbf{w}_h is a nonsingular solution, then there exists a unique solution of (10.14). Consequently, V can be decomposed into the direct sum of two subspaces:

$$V(D) = V_h(D) \oplus \widehat{V}_h(D).$$

This means that for any $\mathbf{w} \in V(D)$, we have

$$\mathbf{w} = P_h\mathbf{w} + P_h^\perp\mathbf{w} = \mathbf{w}_p + \mathbf{w}_q, \quad \mathbf{w}_p \in V_h(D), \quad \mathbf{w}_q \in \widehat{V}_h(D).$$

It is straightforward to show that

$$\begin{cases} \mathcal{A}\mathbf{w}_h(\mathbf{w}_q, \mathbf{v}_p) = 0, & \forall \mathbf{v}_p \in V_h(D), \quad \mathcal{A}\mathbf{w}_h(\mathbf{w}, \mathbf{v}_p) = \mathcal{A}\mathbf{w}_h(\mathbf{w}_p, \mathbf{v}_p), \\ \|\mathbf{w}_q\|_{1,D} \leq Ch^k\|\mathbf{w}\|_{k+1,D}, & \forall \mathbf{w} \in V \cap H^{k+1}(D)^2. \end{cases} \quad (10.15)$$

Next, we present some technical lemmas.

Lemma 10.1. *there exists a constant independent of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ such that*

$$|b_0(\mathbf{u}, \mathbf{w}, \mathbf{v})| \leq C \|\mathbf{u}\|_{0,D}^{\frac{1}{2}} \|\mathbf{u}\|_{1,D}^{\frac{1}{2}} \|\mathbf{w}\|_{1,D} \|\mathbf{v}\|_{0,D}^{\frac{1}{2}} \|\mathbf{v}\|_{1,D}^{\frac{1}{2}}, \quad \forall \mathbf{u}, \mathbf{w}, \mathbf{v} \in V(D). \quad (10.16)$$

Proof: By virtue of Hölder inequality we get

$$|b_0(\mathbf{u}, \mathbf{w}, \mathbf{v})| \leq C \|\mathbf{u}\|_{0,4,D} \|\mathbf{w}\|_{1,D} \|\mathbf{v}\|_{0,4,D},$$

furthermore the Ladyzhenskaya inequality shows that

$$\|\mathbf{u}\|_{0,4,D} \leq C \|\mathbf{u}\|_{0,2,D}^{\frac{1}{2}} \|\mathbf{u}\|_{1,2,D}^{\frac{1}{2}}.$$

Therefore we prove the lemma immediately. \square

Since $\mathbf{w}_0 \in V(D)$, it can be decomposed into $\mathbf{w}_0 = \mathbf{w}_{0p} + \mathbf{w}_{0q}$ with $\mathbf{w}_{0p} \in V_h(D)$, $\mathbf{w}_{0q} \in \widehat{V}_h(D)$.

Lemma 10.2. *The following estimation is valid,*

$$\|\mathbf{w}_{0p} - \mathbf{w}_h\|_{1,D} \leq \frac{2M}{\alpha_0} \|\mathbf{w}_0 - \mathbf{w}_h\|_{1,D}^{\varepsilon_1} \|\mathbf{w}_0 - \mathbf{w}_h\|_{0,D}^{\varepsilon_0}, \quad (10.17)$$

where

$$\varepsilon_1 = \begin{cases} 1, & \text{for the Homogenous Dirichelet B.C.} = (B.C.I), \\ \frac{3}{2}, & \text{for Mixed B.C.} = (B.C.II), \end{cases}$$

$$\varepsilon_0 = \begin{cases} 1, & \text{for the Homogenous Dirichelet B.C.} = (B.C.I), \\ \frac{1}{2}, & \text{for Mixed B.C.} = (B.C.II). \end{cases}$$

Proof: Firstly, equation ((10.1) can be rewritten as

$$\mathcal{A}\mathbf{w}_h(\mathbf{w}_{0q}, \mathbf{v}) + \mathcal{A}\mathbf{w}_h(\mathbf{w}_{0p}, \mathbf{v}) + b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{w}_0 - \mathbf{w}_h, \mathbf{v}) - b_0(\mathbf{w}_h, \mathbf{w}_h, \mathbf{v}) = \langle \mathbf{G}_\eta, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in V(D), \quad (10.18)$$

meanwhile (10.3) as

$$\mathcal{A}\mathbf{w}_h(\mathbf{w}_h, \mathbf{v}) - b_0(\mathbf{w}_h, \mathbf{w}_h, \mathbf{v}) = \langle \mathbf{G}_\eta, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in V_h(D). \quad (10.19)$$

Let $\mathbf{v} \in V_h$, subtracting (10.18) from (10.19) and using (10.15) with $\mathbf{w} = \mathbf{w}_0$, we derive

$$\mathcal{A}\mathbf{w}_h(\mathbf{w}_{0p} - \mathbf{w}_h, \mathbf{v}) = -b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{w}_0 - \mathbf{w}_h, \mathbf{v}). \quad (10.20)$$

Since \mathbf{w}_h is nonsingular, (10.13) shows

$$\frac{1}{2}\alpha_0 \leq \frac{1}{\|\mathbf{w}_{0p} - \mathbf{w}_h\|_{1,D}} \sup_{\mathbf{v} \in V_h(D)} \frac{\mathcal{A}\mathbf{w}_h(\mathbf{w}_{0p} - \mathbf{w}_h, \mathbf{v})}{\|\mathbf{v}\|_{1,D}} = \frac{1}{\|\mathbf{w}_{0p} - \mathbf{w}_h\|_{1,D}} \sup_{\mathbf{v} \in V_h(D)} \frac{-b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{w}_0 - \mathbf{w}_h, \mathbf{v})}{\|\mathbf{v}\|_{1,D}},$$

that is,

$$\frac{1}{2}\alpha_0 \|\mathbf{w}_p - \mathbf{w}_h\|_{1,D} \leq \sup_{\mathbf{v} \in V_h(D)} \frac{|b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{w}_0 - \mathbf{w}_h, \mathbf{v})|}{\|\mathbf{v}\|_{1,D}}.$$

By using (10.16), for any $\mathbf{v} \in V_h(D)$ we get

$$\begin{aligned} |b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{w}_0 - \mathbf{w}_h, \mathbf{v})| &= |b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{v}, \mathbf{w}_0 - \mathbf{w}_h)| \\ &\leq M \|\mathbf{w}_0 - \mathbf{w}_h\|_{0,D} \|\mathbf{w}_0 - \mathbf{w}_h\|_{1,D} \|\mathbf{v}\|_{1,D}, \quad \text{for } B.C.I, \\ |b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{w}_0 - \mathbf{w}_h, \mathbf{v})| &= |b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{v}, \mathbf{w}_0 - \mathbf{w}_h)| \\ &\leq M \|\mathbf{w}_0 - \mathbf{w}_h\|_{0,D}^{\frac{1}{2}} \|\mathbf{w}_0 - \mathbf{w}_h\|_{1,D}^{\frac{3}{2}} \|\mathbf{v}\|_{1,D}, \quad \text{for } B.C.II, \end{aligned} \quad (10.21)$$

Summing up the above relations we draw the conclusion (10.17). \square

Next, let the mapping $\phi(\cdot) : V_h(D) \rightarrow \widehat{V}_h(D)$. we define the manifold \mathcal{M} as the the graph of a function ϕ , that is, $\mathcal{M} = \text{Graph}\phi$, then problem (10.1) can be rewritten as

$$\begin{cases} \text{Find } \phi(\mathbf{w}) \in \widehat{V}_h(D), \text{ such that} \\ \mathcal{A}\mathbf{w}_h(\phi(\mathbf{w}), \mathbf{v}) = b_0(\mathbf{w}, \mathbf{w}, \mathbf{v}) - \mathcal{A}\mathbf{w}_h(\mathbf{w}, \mathbf{v}) - \mathcal{A}\mathbf{w}_h(\mathbf{v}, \mathbf{w}) + \langle \mathbf{G}_\eta, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in \widehat{V}_h. \end{cases} \quad (10.22)$$

We first give an approximation property of the solution of (10.22).

Theorem 10.3. Suppose that the finite element space V_h satisfies the assumptions (H1) \sim (H4). Then there exists a mapping $\phi(\mathbf{w})$ defined by (10.22) which is a Lipschitz continuous function with the Lipschitz constant $l = l(\rho)$, and $\phi(\cdot)$ attracts any solution \mathbf{w}_0 of (10.1), i.e.,

$$\begin{aligned} \text{(H5)} \quad & \|\phi(\mathbf{w}_1) - \phi(\mathbf{w}_2)\|_{1,D} \leq l(\rho) \|\mathbf{w}_1 - \mathbf{w}_2\|_{1,D}, \quad \forall \mathbf{w}_1, \mathbf{w}_2 \in V_h(D) \cap B_\rho, \\ \text{(H6)} \quad & \text{dist}(\mathbf{w}_0, \mathcal{M}) \leq \delta = C(1 + \|\mathbf{w}_{0p}\|_{1,D} + \|\mathbf{w}_h\|_{1,D}) \|\mathbf{w}_0\|_{1,D} h^{2k+\frac{1}{2}}, \end{aligned}$$

where $B_\rho = \{\mathbf{w} | \mathbf{w} \in V(D), \|\mathbf{w}\|_{1,D} \leq \rho\}$.

Proof: Let $\mathbf{w}_i \in V_h$, $\phi_i = \phi(\mathbf{w}_i)$ for $i = 1, 2$, and $\phi = \phi_1 - \phi_2$. If setting $\mathbf{w} = \mathbf{w}_1, \mathbf{w}_2$ in (10.22) respectively and making subtraction, then we get

$$\begin{cases} \mathcal{A}\mathbf{w}_h(\phi, \mathbf{v}) = b_0(\mathbf{w}_1 - \mathbf{w}_2, \mathbf{w}_1, \mathbf{v}) + b_0(\mathbf{w}_2, \mathbf{w}_1 - \mathbf{w}_2, \mathbf{v}) - \mathcal{A}\mathbf{w}_h(\mathbf{w}_1 - \mathbf{w}_2, \mathbf{v}) \\ - \mathcal{A}\mathbf{w}_h(\mathbf{v}, \mathbf{w}_1 - \mathbf{w}_2), \quad \forall \mathbf{v} \in \widehat{V}_h. \end{cases} \quad (10.23)$$

Owing to \mathbf{w}_h is nonsingular and (10.13), the following inequality is valid,

$$\begin{aligned} \frac{1}{2} \alpha_0 \|\phi\|_{1,D} & \leq \sup_{\mathbf{v} \in V_h(D)} \frac{\mathcal{A}\mathbf{w}_h(\phi, \mathbf{v})}{\|\mathbf{v}\|_{1,D}} \leq \sup_{\mathbf{v} \in V(D)} \frac{\mathcal{A}\mathbf{w}_h(\phi, \mathbf{v})}{\|\mathbf{v}\|_{1,D}} \\ & \leq M(\|\mathbf{w}_1\|_{1,D} + \|\mathbf{w}_2\|_{1,D} + \|\mathbf{w}_h\|_{1,D} + 1) \|\mathbf{w}_1 - \mathbf{w}_2\|_{1,D}. \end{aligned}$$

Furthermore by using the triangle inequality $\|\mathbf{w}_h\|_{1,D} \leq \|\mathbf{w}_0 - \mathbf{w}_h\|_{1,D} + \|\mathbf{w}_0\|_{1,D}$, we derive the (H5).

Our task is now to prove (H6). We first know that

$$\begin{aligned} \text{dist}(\mathbf{w}_0, \mathcal{M}) & = \inf_{\mathbf{w} \in \mathcal{M}} \|\mathbf{w}_0 - \mathbf{w}\|_{1,D} \\ & \leq \|\mathbf{w}_0 - (\mathbf{w}_{0p} + \phi(\mathbf{w}_{0p}))\|_{1,D} \\ & = \|\mathbf{w}_0 - \mathbf{w}_{0p} - \phi(\mathbf{w}_{0p})\|_{1,D} \\ & = \|\mathbf{w}_{0q} - \phi(\mathbf{w}_{0p})\|_{1,D}. \end{aligned}$$

For any $\mathbf{v} \in V(D)$, Equation (10.1) can be rewritten as,

$$\mathcal{A}\mathbf{w}_h(\mathbf{w}_{0q}, \mathbf{v}) + \mathcal{A}\mathbf{w}_h(\mathbf{w}_{0p}, \mathbf{v}) + b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{w}_0 - \mathbf{w}_h, \mathbf{v}) - b_0(\mathbf{w}_h, \mathbf{w}_h, \mathbf{v}) = \langle \mathbf{G}_\eta, \mathbf{v} \rangle. \quad (10.24)$$

Choosing $\mathbf{w} = \mathbf{w}_{0p}$ in (10.22) gives

$$\mathcal{A}\mathbf{w}_h(\phi(\mathbf{w}_{0p}), \mathbf{v}) = b_0(\mathbf{w}_{0p}, \mathbf{w}_{0p}, \mathbf{v}) - \mathcal{A}\mathbf{w}_h(\mathbf{w}_{0p}, \mathbf{v}) - \mathcal{A}\mathbf{w}_h(\mathbf{v}, \mathbf{w}_{0p}) + \langle \mathbf{G}_\eta, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in \widehat{V}_h(D). \quad (10.25)$$

Setting $\mathbf{v} \in \widehat{V}_h(D)$ in (10.24) and Subtracting with (10.25). Taking into account (7.15), then we have

$$\begin{aligned} \mathcal{A}\mathbf{w}_h(\mathbf{w}_{0q} - \phi(\mathbf{w}_{0p}), \mathbf{v}) & = -b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{w}_0 - \mathbf{w}_h, \mathbf{v}) + b_0(\mathbf{w}_h, \mathbf{w}_h, \mathbf{v}) - b_0(\mathbf{w}_{0p}, \mathbf{w}_{0p}, \mathbf{v}) \\ & = -b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{w}_0 - \mathbf{w}_h, \mathbf{v}) + b_0(\mathbf{w}_h - \mathbf{w}_{0p}, \mathbf{w}_h, \mathbf{v}) + b_0(\mathbf{w}_{0p}, \mathbf{w}_h - \mathbf{w}_{0p}, \mathbf{v}). \end{aligned}$$

Since \mathbf{w}_0 is a nonsingular solution of (7.1), and noting that (10.5) and (10.15), we get

$$\alpha_0 \|\mathbf{w}_{0q} - \phi(\mathbf{w}_{0p})\|_{1,D} \leq \sup_{\mathbf{v} \in V(D)} \frac{\mathcal{A}\mathbf{w}_0((\mathbf{w}_{0q} - \phi(\mathbf{w}_{0p})), \mathbf{v})}{\|\mathbf{v}\|_{1,D}} \leq I + II.$$

Our problem reduce to the estimation of I, II . It is easy to show that

$$\begin{aligned}
I &= \sup_{\mathbf{v} \in V(D)} \frac{\mathcal{A}\mathbf{w}_h(\mathbf{w}_{0q} - \phi(\mathbf{w}_{0p}), \mathbf{v})}{\|\mathbf{v}\|_{1,D}} \\
&= \sup_{(\mathbf{v}_{0p} + \mathbf{v}_{0q}) \in V(D)} \frac{\mathcal{A}\mathbf{w}_h(\mathbf{w}_{0q} - \phi(\mathbf{w}_{0p}), \mathbf{v}_{0p} + \mathbf{v}_{0q})}{\|\mathbf{v}_{0p} + \mathbf{v}_{0q}\|_{1,D}} \\
&\leq \sup_{\mathbf{v}_{0q} \in \hat{V}_h(D)} \frac{\mathcal{A}\mathbf{w}_h(\mathbf{w}_{0q} - \phi(\mathbf{w}_{0p}), \mathbf{v}_{0q})}{\|\mathbf{v}_{0q}\|_{1,D}} \\
&= \sup_{\mathbf{v}_{0q} \in \hat{V}_h(D)} \frac{-b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{w}_0 - \mathbf{w}_h, \mathbf{v}_{0q}) + b_0(\mathbf{w}_h - \mathbf{w}_{0p}, \mathbf{w}_h, \mathbf{v}_{0q}) + b_0(\mathbf{w}_p, \mathbf{w}_h - \mathbf{w}_{0p}, \mathbf{v}_{0q})}{\|\mathbf{v}_{0q}\|_{1,D}} \\
&\leq M(\|\mathbf{w}_0 - \mathbf{w}_h\|_{0,D}^{\frac{1}{2}} \|\mathbf{w}_0 - \mathbf{w}_h\|_{1,D}^{\frac{3}{2}} + (\|\mathbf{w}_p\|_{1,D} + \|\mathbf{w}_h\|_{1,D}) \|\mathbf{w}_{0p} - \mathbf{w}_h\|_{1,D}) \\
&\leq M(1 + \|\mathbf{w}_{0p}\|_{1,D} + \|\mathbf{w}_h\|_{1,D}) \|\mathbf{w}_0 - \mathbf{w}_h\|_{0,D}^{\frac{1}{2}} \|\mathbf{w}_0 - \mathbf{w}_h\|_{1,D}^{\frac{3}{2}}, \\
II &= \sup_{\mathbf{v} \in V(D)} \frac{\mathcal{A}\mathbf{w}_0(\mathbf{w}_{0q} - \phi(\mathbf{w}_{0p}), \mathbf{v}) - \mathcal{A}\mathbf{w}_h(\mathbf{w}_{0q} - \phi(\mathbf{w}_{0p}), \mathbf{v})}{\|\mathbf{v}\|_{1,D}} \\
&= \sup_{\mathbf{v} \in V(D)} \frac{b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{w}_{0q} - \phi(\mathbf{w}_{0p}), \mathbf{v}) + b_0(\mathbf{w}_{0q} - \phi(\mathbf{w}_{0p}), \mathbf{w}_0 - \mathbf{w}_h, \mathbf{v})}{\|\mathbf{v}\|_{1,D}} \\
&\leq 2M\|\mathbf{w}_0 - \mathbf{w}_h\|_{1,D} \|\mathbf{w}_{0q} - \phi(\mathbf{w}_{0p})\|_{1,D} \\
&\leq 2MCh^k \|\mathbf{w}_{0q} - \phi(\mathbf{w}_{0p})\|_{1,D}.
\end{aligned}$$

where in the fifth step estimation of I we adopt (10.16). Finally, noting that (H1), we have

$$\begin{aligned}
\|\mathbf{w}_{0q} - \phi(\mathbf{w}_{0p})\|_{1,D} &\leq \frac{M(1 + \|\mathbf{w}_{0p}\|_{1,D} + \|\mathbf{w}_h\|_{1,D})}{\alpha_0 - 2MCh^k} \|\mathbf{w}_0 - \mathbf{w}_h\|_{0,D}^{\frac{1}{2}} \|\mathbf{w}_0 - \mathbf{w}_h\|_{1,D}^{\frac{3}{2}} \\
&\leq Ch^{2k + \frac{1}{2}}.
\end{aligned}$$

So we prove (H6). \square

Theorem 10.4. Assume that the assumptions (H1) \sim (H4) for finite element space V_h are satisfied and \mathbf{w}_h is the nonsingular solution of (10.3). Then variational problem (one step Newtonian iteration)

$$\begin{cases} \text{Find } \mathbf{w}_* \in V(D) \text{ such that} \\ \mathcal{A}\mathbf{w}_h(\mathbf{w}_*, \mathbf{v}) = \langle \mathbf{G}_\eta, \mathbf{v} \rangle + b_0(\mathbf{w}_h, \mathbf{w}_h, \mathbf{v}), \quad \forall \mathbf{v} \in V(D), \end{cases} \quad (10.26)$$

has a unique solution \mathbf{w}_* and the following estimation are valid

$$\|\mathbf{w} - \mathbf{w}_*\|_{1,D} \leq Ch^{2k + \varepsilon}, \quad (10.27)$$

where

$$\varepsilon = \begin{cases} 1, & \text{for B.C.I,} \\ \frac{1}{2}, & \text{for B.C.II} \end{cases}$$

Proof: Navier-Stokes equations (10.1) equals to

$$\mathcal{A}\mathbf{w}_h(\mathbf{w}_0, \mathbf{v}) + b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{w}_0 - \mathbf{w}_h, \mathbf{v}) - b_0(\mathbf{w}_h, \mathbf{w}_h, \mathbf{v}) = \langle \mathbf{G}_\eta, \mathbf{v} \rangle, \forall \mathbf{v} \in V(D). \quad (10.28)$$

Subtracting (10.28) from (10.26) leads to

$$\mathcal{A}\mathbf{w}_h(\mathbf{w}_0 - \mathbf{w}_*, \mathbf{v}) + b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{w}_0 - \mathbf{w}_h, \mathbf{v}) = 0, \forall \mathbf{v} \in V(D). \quad (10.29)$$

By applying (10.13) and lemma 10.1 we assert

$$\begin{aligned}
\frac{1}{2}\alpha_0 \|\mathbf{w}_0 - \mathbf{w}_*\|_{1,D} &\leq \sup_{\mathbf{v} \in V_h(D)} \frac{\mathcal{A}\mathbf{w}_h(\mathbf{w}_0 - \mathbf{w}_*, \mathbf{v})}{\|\mathbf{v}\|_{1,D}} \\
&\leq \sup_{\mathbf{v} \in V(D)} \frac{\mathcal{A}\mathbf{w}_h(\mathbf{w}_0 - \mathbf{w}_*, \mathbf{v})}{\|\mathbf{v}\|_{1,D}} \\
&\leq \sup_{\mathbf{v} \in V_h(D)} \frac{-b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{w}_0 - \mathbf{w}_*, \mathbf{v})}{\|\mathbf{v}\|_{1,D}}.
\end{aligned}$$

For B.C.I., we have

$$\begin{aligned} |b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{w}_0 - \mathbf{w}_*, \mathbf{v})| &= |-b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{v}, \mathbf{w}_0 - \mathbf{w}_*)| \\ &\leq M\|\mathbf{w}_0 - \mathbf{w}_h\|_{0,D}\|\mathbf{w}_0 - \mathbf{w}_h\|_{1,D} \\ &\leq \|\mathbf{v}\|_{1,D}MC\|\mathbf{w}_0\|_{k+1,D}^2h^{2k+1}. \end{aligned}$$

Similarly, For B.C. II.,

$$|b_0(\mathbf{w}_0 - \mathbf{w}_h, \mathbf{w}_0 - \mathbf{w}_*, \mathbf{v})| \leq M\|\mathbf{v}\|_{1,D}\|\mathbf{w}_0 - \mathbf{w}_h\|_{0,D}^{\frac{1}{2}}\|\mathbf{w}_0 - \mathbf{w}_h\|_{1,D}^{\frac{3}{2}} \leq MC\|\mathbf{v}\|_{1,D}\|\mathbf{w}_0\|_{k+1,D}^2h^{2k+\frac{1}{2}}.$$

Thus the proof is completed. \square

Remark 10.1. First, it is simple to show that

$$\mathbf{w}_* = \mathbf{w}_h + \phi(\mathbf{w}_h).$$

Second, the variational problem (10.26) is still an infinite dimensional problem. We can apply the standard two-level finite element method on this problem (see Layton et al.[21-23] and references therein).

Theorem 10.5. Suppose that the assumptions in theorem 10.4 are satisfied. V_{h^*} is a finite element subspace with mesh parameter $h^* \leq h$ and satisfies assumptions (H1) \sim (H4) with integer $m \leq k$. If (\mathbf{w}_{h^*}) is a Galerkin finite element approximation solution to (10.26), that is

$$\begin{cases} \text{Find } (\mathbf{w}_{h^*}) \in V_{h^*} \text{ such that} \\ \mathcal{A}\mathbf{w}_{h^*}(\mathbf{w}_{h^*}, \mathbf{v}) = (\mathbf{G}_\eta, \mathbf{v}) + b_0(\mathbf{w}_h, \mathbf{w}_h, \mathbf{v}) \quad \forall \mathbf{v} \in V_{h^*}. \end{cases} \quad (10.30)$$

Then the following error estimation holds

$$\|\mathbf{w}_* - \mathbf{w}_{h^*}\|_{1,D} \leq Ch^{*(m+1)}(\|\mathbf{w}_*\|_{m+1}). \quad (10.31)$$

Proof: The proof is omitted. \square

Combining theorem 10.4 and 10.5 leads to our final conclusion,

Theorem 10.6. Suppose that the assumptions in theorem 10.4 and 10.5 are satisfied. Then we have following estimation

$$\|(\mathbf{w}_0 - \mathbf{w}_{h^*})\|_{1,D} \leq C(h^{2k+\varepsilon} + h^{*(m+1)}).$$

In particular, if choosing $h^* = h^{(2k+1)/(m+1)}$, then we have

$$\|(\mathbf{w}_0 - \mathbf{w}_{h^*})\|_{1,D} \leq C(h^{2k+\varepsilon}).$$

Proof: The proof is omitted. \square

Algorithm 1. Here we present the finite element approximation algorithm based on the approximate inertial manifold, i.e.,

- **Step1:** Solve the nonlinear problem (10.3) on the coarse grid with mesh size h ,
- **Step2:** Solve the linear problem (10.30) on the fine grid with mesh size h^* .

Remark 10.2. If we use the linear finite element method for (10.3) and (10.30), respectively, then on the two dimensional problem the following estimates hold

$$\|(\mathbf{w}_0 - \mathbf{w}_{h^*})\|_{1,D} \leq ch^3 \approx ch^{*^2},$$

where $h^* \approx h^{\frac{3}{2}}$. As we know that in Layton[19, Theorem 2], the result is

$$\|(\mathbf{w}_0 - \mathbf{w}_{h^*})\|_{1,D} \leq ch^2 \approx ch^*,$$

with $h^* \approx h^2$. This shows that our results is much better than that in [19].

A Appendix

This section gathers most of the preliminary knowledge that will be required in this article. In subsection 1, we focus on the the expressions of some physical and geometrical quantities in the new coordinates system; then in subsection 2, The Navier-Stokes Equation in the new coordinate system is derived. Finally, we consider the Gâteaux derivative of the solutions of the Navier-Stokes equations with respect to the shape of blade.

A.1 Some Physical and Geometrical Quantities

In order to simplicity, we consider the 3D fluid flow in an flow passage in an impeller with rotating angular velocity $\boldsymbol{\omega} = (0, 0, \omega)$ around its axis, and the thickness of the blade is uniform. Let (x, y, z) be the cartesian coordinate system out of the impeller in the Euclidean space R^3 , and three coordinate basis vector are $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively. Furthermore, (r, θ, z) be the cylindrical coordinate system attached to and fixed on the impeller, and $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ are three basis vector of this system respectively. Next we define a new coordinate system (x^1, x^2, x^3) through the following relations

$$\begin{cases} x^1 = z, & x^2 = r, & x^3 = \xi = \varepsilon^{-1}(\theta - \Theta(x^1, x^2)), \\ r = x^2, & z = x^1, & \theta = \varepsilon\xi + \Theta(x^1, x^2), \end{cases} \quad (A.I.1)$$

where $\Theta(x^1, x^2)$ be a smooth mapping from a bounded smooth enough subset $D \subset R^2$ into R , especially, $(x^1, x^2, \Theta(x^1, x^2))$ denote an arbitrary point on the blade surface. Let \mathbf{e}_i be the basic vectors of this new coordinate system. The parameter ξ satisfies $0 \leq \xi \leq 1$ and obviously, $\xi = \text{const}$ represent a surface \mathfrak{S}_ξ in R^3 , which can be obtained by a rotation of the blade through an angle of $\varepsilon\xi$ degree.

Next, we present the following proposition.

Proposition A.1. *The covariant components $a_{\alpha\beta}$ and contra-variant components $a^{\alpha\beta}$ of the metric tensor of the surface \mathfrak{S}_ξ and the covariant components g_{ij} and the contra-variant components g^{ij} of the metric tensor of 3D Euclidean space \mathbf{R}^3 are given by, respectively,*

$$\begin{cases} a_{\alpha\beta} = \delta_{\alpha\beta} + (x^2)^2 \Theta_\alpha \Theta_\beta, & a = \det(a_{\alpha\beta}) = 1 + (x^2)^2 (\Theta_1^2 + \Theta_2^2), & \Theta_\alpha = \frac{\partial \Theta}{\partial x^\alpha}, \\ a^{\alpha\beta} a_{\beta\sigma} = \delta_\sigma^\alpha, & a^{11} = \frac{a_{22}}{\sqrt{a}}, & a^{22} = \frac{a_{11}}{\sqrt{a}}, & a^{12} = a^{21} = -\frac{a_{12}}{\sqrt{a}}, \\ g_{\alpha\beta} = a_{\alpha\beta}, & g_{3\alpha} = g_{\alpha 3} = \varepsilon r^2 \Theta_\alpha, & g_{33} = \varepsilon^2 r^2, & g = \det(g_{ij}) = \varepsilon^2 r^2, \\ g^{ij} g_{jk} = \delta_k^i, & g_{\alpha\beta} = a_{\alpha\beta}, & g_{3\alpha} = g_{\alpha 3} = \varepsilon r^2 \Theta_\alpha, & g_{33} = \varepsilon^2 r^2, \end{cases} \quad (A.I.2)$$

where δ_j^i is the Kronecker symbol. Furthermore, we have the followings conclusions

1. Rotating Angular Velocity $\boldsymbol{\omega}$

$$\begin{cases} \boldsymbol{\omega} = \omega \mathbf{e}_1 - \omega \varepsilon^{-1} \Theta_1 \mathbf{e}_3 \\ \omega^1 = \omega, & \omega^2 = 0, & \omega^3 = -\omega \varepsilon^{-1} \Theta_1, \end{cases} \quad (A.I.3)$$

2. Coriolis Forces

$$\begin{cases} 2\boldsymbol{\omega} \times \mathbf{w} = C^1 \mathbf{e}_1 + C^2 \mathbf{e}_2 + C^3 \mathbf{e}_3 \\ C^1 = 0, & C^2 = -2r\omega \Pi(\mathbf{w}, \Theta), \\ C^3 = 2\omega \varepsilon^{-1} (r\Theta_2 \Pi(\mathbf{w}, \Theta) + \frac{w^2}{r}), \end{cases} \quad (A.I.4)$$

3. Unite Normal vector to the Surface \mathfrak{S}_ξ

$$\begin{cases} \mathbf{n} = -x^2 \Theta_\alpha / \sqrt{a} \mathbf{e}_\alpha + (\varepsilon x^2)^{-1} \frac{1+r^2 \Theta_2^2}{\sqrt{a}} \mathbf{e}_3, \\ n^\alpha = -x^2 \Theta_\alpha / \sqrt{a}, & n^3 = (\varepsilon x^2)^{-1} \frac{1+r^2 \Theta_2^2}{\sqrt{a}}. \end{cases} \quad (A.I.5)$$

4. Curvature Tensor of the Surface \mathfrak{S}_ξ (Second Fundamental Form)

$$\begin{aligned} b_{11} &= \frac{1}{\sqrt{a}} (\Theta_2 (a_{11} - 1) + x^2 \Theta_{11}), & b_{12} &= b_{21} = \frac{1}{\sqrt{a}} (\Theta_1 a_{12} + x^2 \Theta_{12}), \\ b_{22} &= \frac{1}{\sqrt{a}} (\Theta_2 (a_{22} + 1) + x^2 \Theta_{22}), & b &= \det(b_{\alpha\beta}) = b_{11} b_{22} - b_{12}^2, \end{aligned} \quad (A.I.6)$$

where

$$|\tilde{\nabla}\Theta|^2 = \Theta_1^2 + \Theta_2^2, \quad \Delta\Theta = a^{\alpha\beta}\Theta_{\alpha\beta}, \quad \tilde{\Delta}\Theta = \Theta_{11} + \Theta_{22}, \quad \Theta_{\alpha\beta} = \partial_\alpha\partial_\beta\Theta, \quad (A.I.7)$$

5. Mean Curvature and Gaussian Curvature of the Surface \mathfrak{S}_ξ

$$\begin{cases} K = b/a, \\ 2H = \frac{1}{a\sqrt{a}}[x^2(a_{22}\Theta_{11} + a_{11}\Theta_{22}) - 2a_{12}\Theta_{12})\Theta_2(2a_{11}a_{22} + a_{11} - a_{22}) - 2\Theta_1a_{12}^2, \end{cases} \quad (A.I.8)$$

Proof: Firstly, from (A.I.1) we have

$$\begin{cases} x = x(x^1, x^2, \xi) = r \cos \theta = x^2 \cos(\varepsilon\xi + \Theta(x^1, x^2)) \\ y = y(x^1, x^2, \xi) = r \sin \theta = x^2 \sin(\varepsilon\xi + \Theta(x^1, x^2)) \\ z = z(x^1, x^2, \xi) = x^1 \end{cases} \quad (A.I.9)$$

Therefore,

$$\begin{cases} \frac{\partial x}{\partial x^1} = -x^2 \sin \theta \Theta_1, & \frac{\partial x}{\partial x^2} = \cos \theta - x^2 \sin \theta \Theta_2, & \frac{\partial x}{\partial \xi} = -x^2 \sin \theta \varepsilon, \\ \frac{\partial y}{\partial x^1} = x^2 \cos \theta \Theta_1, & \frac{\partial y}{\partial x^2} = \sin \theta + x^2 \cos \theta \Theta_2, & \frac{\partial y}{\partial \xi} = x^2 \cos \theta \varepsilon, \\ \frac{\partial z}{\partial x^1} = 1, & \frac{\partial z}{\partial x^2} = \frac{\partial z}{\partial x^3} = 0, \end{cases} \quad (A.I.10)$$

where $\theta = \varepsilon\xi + \Theta(x^1, x^2)$. From (A.I.10) we get

$$\frac{\partial(x, y, z)}{\partial(x^1, x^2, x^3)} = \varepsilon x^2. \quad (A.I.11)$$

It is well known that

$$\begin{cases} \mathbf{e}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, & \mathbf{e}_\theta = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}, \\ \mathbf{i} = \cos \theta \mathbf{e}_r - \sin \theta \mathbf{e}_\theta, & \mathbf{j} = \sin \theta \mathbf{e}_r + \cos \theta \mathbf{e}_\theta, \end{cases} \quad (A.I.12)$$

Let $\mathbf{R}(x^1, x^2, \xi) = x(x^1, x^2, \xi)\mathbf{i} + y(x^1, x^2, \xi)\mathbf{j} + z(x^1, x^2, \xi)\mathbf{k}$ denote the radial vector at the point $P = (x^1, x^2, \xi)$. Then the covariant basic vectors $(\mathbf{e}_\alpha, \mathbf{e}_3)$ and the contra-variant basic vectors $(\mathbf{e}^\alpha, \mathbf{e}^3)$ in the new curvilinear coordinate system (x^α, ξ) are given by, respectively,

$$\begin{cases} \mathbf{e}_\alpha = \partial_\alpha \mathbf{R} = \partial_\alpha x \mathbf{i} + \partial_\alpha y \mathbf{j} + \partial_\alpha z \mathbf{k}, & \mathbf{e}_3 = \frac{\partial}{\partial \xi} \mathbf{R} = \frac{\partial x}{\partial \xi} \mathbf{i} + \frac{\partial y}{\partial \xi} \mathbf{j} + \frac{\partial z}{\partial \xi} \mathbf{k}, \\ \mathbf{e}_1 = x^2 \Theta_1 \mathbf{e}_\theta + \mathbf{k} = -x^2 \sin \theta \Theta_1 \mathbf{i} + x^2 \cos \theta \Theta_1 \mathbf{j} + \mathbf{k}, \\ \mathbf{e}_2 = \Theta_2 x^2 \mathbf{e}_\theta + \mathbf{e}_r = (\cos \theta - x^2 \sin \theta \Theta_2) \mathbf{i} + (\sin \theta + x^2 \cos \theta \Theta_2) \mathbf{j}, \\ \mathbf{e}_3 = x^2 \varepsilon \mathbf{e}_\theta = -\varepsilon x^2 \sin \theta \mathbf{i} + \varepsilon x^2 \cos \theta \mathbf{j}, \\ \mathbf{e}^i = g^{ij} \mathbf{e}_j, & \mathbf{e}^\alpha = \mathbf{e}_\alpha - \varepsilon^{-1} \Theta_\alpha \mathbf{e}_3, & \mathbf{e}^3 = -\varepsilon^{-1} \Theta_\alpha \mathbf{e}_\alpha + (r\varepsilon)^{-2} a \mathbf{e}_3, \\ \mathbf{e}^1 = \mathbf{k}, & \mathbf{e}^2 = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, \\ \mathbf{e}^3 = -(r\varepsilon)^{-1} (\sin \theta + r\Theta_2 \cos \theta) \mathbf{i} + (r\varepsilon)^{-1} (\cos \theta - r\Theta_2 \sin \theta) \mathbf{j} - \varepsilon^{-1} \Theta_1 \mathbf{k}. \end{cases} \quad (A.I.13)$$

Inversely, we have

$$\begin{cases} \mathbf{e}_r = \mathbf{e}_2 - \varepsilon^{-1} \Theta_2 \mathbf{e}_3, & \mathbf{e}_\theta = (\varepsilon x^2)^{-1} \mathbf{e}_3, & \mathbf{k} = \mathbf{e}_1 - \varepsilon^{-1} \Theta_1 \mathbf{e}_3, \\ \mathbf{i} = \cos \theta \mathbf{e}_2 - (\varepsilon^{-1} \cos \theta \Theta_2 + (\varepsilon x^2)^{-1} \sin \theta) \mathbf{e}_3, \\ \mathbf{j} = \sin \theta \mathbf{e}_2 + ((\varepsilon x^2)^{-1} \cos \theta - \varepsilon^{-1} \Theta_2 \sin \theta) \mathbf{e}_3, \end{cases} \quad (A.I.14)$$

For any fixed ξ , the mapping

$$\mathbf{R}(x^1, x^2; \xi) = x(x^1, x^2, \xi)\mathbf{i} + y(x^1, x^2, \xi)\mathbf{j} + z(x^1, x^2, \xi)\mathbf{k}$$

define a 2-dimensional surface \mathfrak{S}_ξ with single parameter ξ and the covariant components of metric tensor of \mathfrak{S}_ξ is expressed as

$$a_{\alpha\beta} = \mathbf{e}_\alpha \mathbf{e}_\beta = \delta_{\alpha\beta} + (x^2)^2 \Theta_\alpha \Theta_\beta, \quad a = \det(a_{\alpha\beta}) = 1 + (x^2)^2 (\Theta_1^2 + \Theta_2^2) = 1 + |\tilde{\nabla}\Theta|^2.$$

Meantime, the covariant components of the metric tensor of 3D Euclidean space R^3 in the coordinate system (x^1, x^2, ξ) are given by

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$$

From this and the expression (A.I.13) it is easy to derive (A.I.1).

Next, we consider the angular velocity vector. Obviously,

$$\begin{cases} \boldsymbol{\omega} = \omega \mathbf{k} = \omega \mathbf{e}_1 - \omega \varepsilon^{-1} \Theta_1 \mathbf{e}_3, \\ \omega^1 = \omega, \quad \omega^2 = 0, \quad \omega^3 = -\varepsilon^{-1} \omega \Theta_1, \end{cases} \quad (A.I.15)$$

which norm can be calculated as

$$\begin{aligned} |\boldsymbol{\omega}|^2 &= \boldsymbol{\omega} \cdot \boldsymbol{\omega} = g_{ij} \omega^i \omega^j = g_{11} \omega^1 \omega^1 + g_{33} \omega^3 \omega^3 + 2g_{13} \omega^1 \omega^3 \\ &= a_{11}(\omega)^2 + r^2 \varepsilon^2 (\omega)^2 \varepsilon^{-2} \Theta_1^2 + 2\varepsilon r^2 \Theta_1 (\omega)^2 (-\varepsilon^{-1} \Theta_1) \\ &= (\omega)^2 (a_{11} + r^2 \Theta_1^2 - 2r^2 \Theta_1^2) = (\omega)^2. \end{aligned}$$

Similarly, from the coordinate relation (A.I.1) and the definition of ε_{ijk} , the Coriolis force is formulated as

$$\begin{aligned} \mathbf{C} &= 2\boldsymbol{\omega} \times \mathbf{w} = 2\varepsilon_{ijk} \omega^j w^k \mathbf{e}^i = (2\varepsilon_{ijk} \omega^j w^k) g^{im} \mathbf{e}_m \\ &= 2\varepsilon_{ijk} \omega^j w^k (g^{i1} \mathbf{e}_1 + g^{i2} \mathbf{e}_2 + g^{i3} \mathbf{e}_3) = C^i \mathbf{e}_i, \\ C^1 &= 2(g^{i1} \varepsilon_{i1k} \omega w^k + g^{i1} \varepsilon_{i3k} (-\omega \varepsilon^{-1} \Theta_1) w^k) \\ &= 2\omega (g^{\alpha 1} \varepsilon_{\alpha 13} w^3 + g^{31} \varepsilon_{312} w^2 - g^{\alpha 1} \varepsilon_{\alpha 3\beta} \varepsilon^{-1} \Theta_1 w^\beta) \\ &= 2\omega (0 - \varepsilon^{-1} \Theta_1 w^2 \sqrt{g} - \varepsilon_{132} \varepsilon^{-1} \Theta_1 w^2) = 0, \\ C^2 &= 2(g^{i2} \varepsilon_{i1k} \omega w^k + g^{i2} \varepsilon_{i3k} (-\omega \varepsilon^{-1} \Theta_1) w^k) \\ &= 2\omega (g^{\alpha 2} \varepsilon_{\alpha 13} w^3 + g^{32} \varepsilon_{312} w^2 - \varepsilon^{-1} \Theta_1 g^{\alpha 2} \varepsilon_{\alpha 3\beta} w^\beta) \\ &= 2\omega (\varepsilon_{213} w^3 - \varepsilon^{-1} \Theta_2 w^2 \varepsilon_{312} - \varepsilon^{-1} \Theta_1 \varepsilon_{231} w^1) \\ &= 2\omega \sqrt{g} (-w^3 - \varepsilon^{-1} \Theta_2 w^2 - \varepsilon^{-1} \Theta_1 w^1) = -2r\omega \Pi(w, \Theta), \\ C^3 &= 2(g^{i3} \varepsilon_{i1k} \omega w^k + g^{i3} \varepsilon_{i3k} (-\omega \varepsilon^{-1} \Theta_1) w^k) \\ &= 2(g^{\alpha 3} \varepsilon_{\alpha 13} \omega w^3 + g^{33} \varepsilon_{312} \omega w^2 + g^{\alpha 3} \varepsilon_{\alpha 3\beta} (-\omega \varepsilon^{-1} \Theta_1) w^\beta) \\ &= 2\omega (-\varepsilon^{-1} \Theta_2 \varepsilon_{213} w^3 + g^{33} \varepsilon_{312} w^2 - \varepsilon^{-1} \Theta_1 (-\varepsilon^{-1} \Theta_2 \varepsilon_{231} w^1 - \varepsilon^{-1} \Theta_1 \varepsilon_{132} w^2)) \\ &= 2\omega \sqrt{g} (\varepsilon^{-1} \Theta_2 w^3 + g^{33} w^2 + \varepsilon^{-2} \Theta_1 (\Theta_2 w^1 - \Theta_1 w^2)). \end{aligned}$$

Owing to the identity

$$g^{33} = (r\varepsilon)^{-2} w^2 + \varepsilon^{-2} (\Theta_1^2 + \Theta_2^2) w^2,$$

we have

$$C^3 = 2r\omega \varepsilon^{-1} \Theta_2 \Pi(w, \Theta) + 2\omega (r\varepsilon)^{-1} w^2.$$

where

$$\Pi(w, \Theta) = \varepsilon w^3 + \Theta_\alpha w^\alpha.$$

Therefore,

$$2\boldsymbol{\omega} \times \mathbf{w} = -2r\omega \Pi(w, \Theta) \mathbf{e}_2 + (2r\omega \varepsilon^{-1} \Theta_2 \Pi(w, \Theta) + 2\omega (r\varepsilon)^{-1} w^2) \mathbf{e}_3]$$

Finally, the contravariant components of Coriolis force in the new coordinate system is given by

$$C^1 = 0, \quad C^2 = -2\omega r \Pi(w, \Theta), \quad C^3 = 2\omega (\varepsilon)^{-1} (r\Theta_2 \Pi(w, \Theta) + \frac{w^2}{r}). \quad (A.I.16)$$

Next we consider the unite normal vector \mathbf{n} of \mathfrak{S}_ξ . At first, it is well know that,

$$\mathbf{n} = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|} = \frac{1}{\sqrt{a}} (\mathbf{e}_1 \times \mathbf{e}_2) = n_x \mathbf{i} + n_y \mathbf{j} + n_z \mathbf{k} = n^i \mathbf{e}_i$$

By virtue of (A.I.13) and (A.I.14), the above expression shows that

$$\begin{aligned}
\mathbf{e}_1 \times \mathbf{e}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ (\mathbf{e}_1)_x & (\mathbf{e}_1)_y & (\mathbf{e}_1)_z \\ (\mathbf{e}_2)_x & (\mathbf{e}_2)_y & (\mathbf{e}_2)_z \end{vmatrix} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\Theta_1 \sin \theta & r\Theta_1 \cos \theta & 1 \\ \cos \theta - r\Theta_2 \sin \theta & \sin \theta + r\Theta_2 \cos \theta & 0 \end{vmatrix} \\
&= -(\sin \theta + r\Theta_2 \cos \theta)\mathbf{i} + (\cos \theta - r\Theta_2 \sin \theta)\mathbf{j} - r\Theta_1\mathbf{k}
\end{aligned}$$

From this we obtain the contra-variant components of \mathbf{n} in the cartesian and the new coordinate system,

$$\begin{cases} n_x = -\frac{1}{\sqrt{a}}(\sin \theta + x^2\Theta_2 \cos \theta), & n_y = \frac{1}{\sqrt{a}}(\cos \theta - x^2\Theta_2 \sin \theta), \\ n_z = -x^2\Theta_1/\sqrt{a}. \\ n^\alpha = -x^2\Theta_\alpha/\sqrt{a}, & n^3 = (r\varepsilon)^{-1}\sqrt{a}, \end{cases} \quad (A.I.17)$$

Now we calculate the curvature tensor of the surface \mathfrak{S}_ξ . Noting that

$$b_{\alpha\beta} = -\frac{1}{2}(\mathbf{n}_\alpha \mathbf{e}_\beta + \mathbf{n}_\beta \mathbf{e}_\alpha) = \mathbf{n} \mathbf{e}_{\alpha\beta} = \frac{1}{\sqrt{a}} \mathbf{e}_1 \times \mathbf{e}_2 \cdot \mathbf{e}_{\alpha\beta}, \quad (A.I.18)$$

If the radial vector at the point P on \mathfrak{S}_ξ is denoted by \mathfrak{R} , then

$$\mathbf{e}_{\alpha\beta} = \frac{\partial^2 \mathfrak{R}}{\partial x^\alpha \partial x^\beta} = x_{\alpha\beta} \mathbf{i} + y_{\alpha\beta} \mathbf{j} + z_{\alpha\beta} \mathbf{k},$$

where $x_{\alpha\beta} = \partial_\alpha \partial_\beta x$. Therefore

$$\begin{aligned}
\sqrt{a} b_{\alpha\beta} &= \begin{vmatrix} x_{\alpha\beta} & y_{\alpha\beta} & z_{\alpha\beta} \\ (\mathbf{e}_1)_x & (\mathbf{e}_1)_y & (\mathbf{e}_1)_z \\ (\mathbf{e}_2)_x & (\mathbf{e}_2)_y & (\mathbf{e}_2)_z \end{vmatrix} = \begin{vmatrix} x_{\alpha\beta} & y_{\alpha\beta} & 0 \\ -r\Theta_1 \sin \theta & r\Theta_1 \cos \theta & 1 \\ \cos \theta - r\Theta_2 \sin \theta & \sin \theta + r\Theta_2 \cos \theta & 0 \end{vmatrix} \\
&= -[(x_{\alpha\beta} \sin \theta - y_{\alpha\beta} \cos \theta) + r\Theta_2(x_{\alpha\beta} \cos \theta + y_{\alpha\beta} \sin \theta)] \\
&= -[(x_{\alpha\beta} + r\Theta_2 y_{\alpha\beta}) \sin \theta + (r\Theta_2 x_{\alpha\beta} - y_{\alpha\beta}) \cos \theta],
\end{aligned} \quad (A.I.19)$$

Simply calculation from (A.I.8-A.I.10) shows that

$$\begin{aligned}
x_{11} &= \frac{\partial^2 x}{\partial (x^1)^2} = -x^2(\Theta_{11} \sin \theta + \Theta_1^2 \cos \theta), \\
x_{12} &= -\Theta_1 \sin \theta - x^2(\Theta_{12} \sin \theta + \Theta_1 \Theta_2 \cos \theta), \\
x_{22} &= -2\Theta_2 \sin \theta - x^2(\Theta_{22} \sin \theta + \Theta_2^2 \cos \theta), \\
y_{11} &= x^2(\Theta_{11} \cos \theta - \Theta_1^2 \sin \theta), \quad y_{12} = \Theta_1 \cos \theta + x^2(\Theta_{12} \cos \theta - \Theta_1 \Theta_2 \sin \theta), \\
y_{22} &= 2\Theta_2 \cos \theta + x^2(\Theta_{22} \cos \theta - \Theta_2^2 \sin \theta), \\
z_{\alpha\beta} &= 0,
\end{aligned} \quad (A.I.20)$$

Substituting (A.I.20) into (A.I.19) leads to

$$\begin{aligned}
b_{11} &= \frac{1}{\sqrt{a}}(\Theta_2(a_{11} - 1) + x^2\Theta_{11}), \quad b_{12} = b_{21} = \frac{1}{\sqrt{a}}(\Theta_1 a_{12} + x^2\Theta_{12}), \\
b_{22} &= \frac{1}{\sqrt{a}}(\Theta_2(a_{22} + 1) + x^2\Theta_{22}), \quad b = \det(b_{\alpha\beta}) = b_{11}b_{22} - b_{12}^2,
\end{aligned} \quad (A.I.21)$$

Finally, the mean curvature and the Gaussian curvature are calculated as

$$\begin{cases} K &= \frac{b}{a}, \\ 2H &= \frac{1}{a\sqrt{a}}[x^2(a_{22}\Theta_{11} + a_{11}\Theta_{22}) - 2a_{12}\Theta_{12})\Theta_2(2a_{11}a_{22} + a_{11} - a_{22}) - 2\Theta_1 a_{12}^2], \end{cases} \quad (A.I.22)$$

Those are (A.I.4)(A.I.5). \square

In the next place, we consider the Christoffel symbols and the covariant derivatives under the new coordinate system.

Proposition A.2. *Under the new curvilinear coordinate system (x^α, ξ) , the Christoffel symbols and covariant derivatives are respectively given by*

$$\begin{cases} \Gamma_{\beta\gamma}^\alpha = -r\delta_{2\alpha}\Theta_\beta\Theta_\gamma, & \Gamma_{3\beta}^\alpha = -\varepsilon r\delta_{2\alpha}\Theta_\beta, \\ \Gamma_{\alpha\beta}^3 = \varepsilon^{-1}r^{-1}(\delta_{2\alpha}\delta_\beta^\lambda + \delta_{2\beta}\delta_\alpha^\lambda)\Theta_\lambda + \varepsilon^{-1}\Theta_{\alpha\beta} + \varepsilon^{-1}r\Theta_2\Theta_\alpha\Theta_\beta, \\ \Gamma_{3\alpha}^3 = \Gamma_{\alpha 3}^3 = r^{-1}\delta_{2\alpha} + r\Theta_2\Theta_\alpha & \Gamma_{33}^\alpha = -\varepsilon^2 r\delta_{2\alpha}, \quad \Gamma_{33}^3 = \varepsilon r\Theta_2, \end{cases} \quad (A.I.23)$$

and

$$\begin{cases} \nabla_\alpha w^\beta = \frac{\partial w^\beta}{\partial x^\alpha} - r\delta_2^\beta \Theta_\alpha \Pi(w, \Theta), \\ \nabla_\alpha w^3 = \frac{\partial w^3}{\partial x^\alpha} + \varepsilon^{-1}(x^2)^{-1}w^2\Theta_\alpha + \varepsilon^{-1}w^\beta\Theta_{\alpha\beta} + (\varepsilon x^2)^{-1}a_{2\alpha}\Pi(w, \Theta), \\ \nabla_3 w^\alpha = \frac{\partial w^\alpha}{\partial \xi} - x^2\varepsilon\delta_{2\alpha}\Pi(w, \Theta), \quad \nabla_3 w^3 = \frac{\partial w^3}{\partial \xi} + \frac{w^2}{x^2} + x^2\Theta_2\Pi(w, \Theta), \\ \text{div} \mathbf{w} = \frac{\partial w^\alpha}{\partial x^\alpha} + \frac{w^2}{x^2} + \frac{\partial w^3}{\partial \xi}, \quad \Pi(w, \Theta) = \varepsilon w^3 + w^\beta\Theta_\beta. \end{cases} \quad (A.I.24)$$

Proof: From (A.I.13) it follows that

$$\begin{aligned} \mathbf{e}_{ij} &= \partial_i \mathbf{e}_j, \\ \mathbf{e}_{11} &= -x^2(\cos \theta \Theta_1^2 + \sin \theta \Theta_{11})\mathbf{i} + x^2(-\sin \theta \Theta_1^2 + \cos \theta \Theta_{11})\mathbf{j}, \\ \mathbf{e}_{12} &= \mathbf{e}_{21} = (-\sin \theta \Theta_1 - x^2(\cos \theta \Theta_1 \Theta_2 + \sin \theta \Theta_{12}))\mathbf{i} \\ &\quad + (\cos \theta \Theta_1 + x^2(-\sin \theta \Theta_1 \Theta_2 + \cos \theta \Theta_{12}))\mathbf{j}, \\ \mathbf{e}_{22} &= (-2\sin \theta \Theta_2 - x^2(\cos \theta \Theta_2 \Theta_2 + \sin \theta \Theta_{22}))\mathbf{i} + (2\cos \theta \Theta_1 + x^2(-\sin \theta \Theta_2 \Theta_2 + \cos \theta \Theta_{22}))\mathbf{j}, \\ \mathbf{e}_{13} &= \mathbf{e}_{31} = -(r\varepsilon)\Theta_1(\cos \theta \mathbf{i} + \sin \theta \mathbf{j}), \quad \mathbf{e}_{33} = -r\varepsilon^2(\cos \theta \mathbf{i} + \sin \theta \mathbf{j}), \\ \mathbf{e}_{23} &= \mathbf{e}_{32} = -\varepsilon(\sin \theta + r\Theta_2 \cos \theta)\mathbf{i} + \varepsilon(\cos \theta - r\Theta_2 \sin \theta)\mathbf{j}, \\ \Gamma_{jk}^i &= \mathbf{e}^i \mathbf{e}_{jk}, \\ \Gamma_{11}^1 &= \mathbf{e}^1 \mathbf{e}_{11} = k\mathbf{e}_{11} = 0, \quad \Gamma_{11}^2 = -x^2\Theta_1^2, \quad \Gamma_{11}^3 = (\varepsilon)^{-1}(r\Theta_2\Theta_1^2 + \Theta_{11}), \\ \Gamma_{12}^1 &= \Gamma_{21}^1 = 0, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = -r\Theta_1\Theta_2, \quad \Gamma_{12}^3 = \Gamma_{21}^3 = (r\varepsilon)^{-1}(\Theta_1 a_{22} + r\Theta_{12}), \\ \Gamma_{22}^1 &= 0, \quad \Gamma_{22}^2 = -r\Theta_2^2, \quad \Gamma_{22}^3 = (r\varepsilon)^{-1}[2\Theta_2 + r(\Theta_{22} + r\Theta_2^2)] = (r\varepsilon)^{-1}[\Theta_2(1 + a_{22}) + r\Theta_{22}], \\ \Gamma_{13}^1 &= \Gamma_{31}^1 = 0, \quad \Gamma_{13}^2 = \Gamma_{31}^2 = -(r\varepsilon)^{-1}\Theta_1, \quad \Gamma_{13}^3 = \Gamma_{31}^3 = r\Theta_1\Theta_2, \\ \Gamma_{23}^1 &= \Gamma_{32}^1 = 0, \quad \Gamma_{23}^2 = \Gamma_{32}^2 = -r\varepsilon\Theta_2, \quad \Gamma_{23}^3 = \Gamma_{32}^3 = r^{-1}a_{22}, \\ \Gamma_{33}^1 &= 0, \quad \Gamma_{33}^2 = -r\varepsilon^2, \quad \Gamma_{33}^3 = -r\varepsilon\Theta_2, \end{aligned}$$

This yields (A.I.23). The (A.I.24) can be obtain from (A.I.23) and

$$\nabla_i w^j = \partial_i w^j + \Gamma_{ik}^j w^k.$$

This ends the proof. \square

A.2 The Navier-Stokes Equation In the New Coordinate System

Proposition A.3. *The Rotating Navier-stokes equations in the new coordinate system (x^α, ξ) can be written as*

$$\begin{cases} \frac{\partial w^\alpha}{\partial x^\alpha} + \frac{\partial w^3}{\partial \xi} + \frac{w^2}{r} = 0, \\ \mathcal{N}^\alpha(\mathbf{w}, p, \Theta) := -\nu \tilde{\Delta} w^\alpha + \nabla_\alpha p + C^\alpha(\mathbf{w}, \omega) - \nu l^\alpha(\mathbf{w}, \Theta) + \frac{\partial}{\partial \xi}(-\nu l_\xi^\alpha(\mathbf{w}, \Theta) \\ \quad - \varepsilon^{-1}\Theta_\alpha p) + N_x^\alpha(\mathbf{w}, p) + \mathcal{N}_\xi^\alpha(\mathbf{w}, p) = f^\alpha, \\ \mathcal{N}^3(\mathbf{w}, p, \Theta) := -\nu \tilde{\Delta} w^3 - \varepsilon^{-1}\Theta_\alpha \frac{\partial p}{\partial x^\alpha} + C^3(\mathbf{w}, \omega) - \nu l^3(\mathbf{w}, \Theta) \\ \quad + \frac{\partial}{\partial \xi}(-\nu l_\xi^3(\mathbf{w}, \Theta) + (r\varepsilon)^{-2}ap) + N_x^3(\mathbf{w}, p) + \mathcal{N}_\xi^3(\mathbf{w}, p) = f^3, \end{cases} \quad (A.II.1)$$

where $\mathbf{C}(\mathbf{w}, \omega)$ is Coriolis forces defined in (A.I.4), and the other shortening symbols are definitely expressed as, respectively,

$$\begin{cases} N_x^\alpha(\mathbf{w}, \mathbf{w}) = w^\beta \frac{\partial w^\alpha}{\partial x^\beta} - r\delta_{2\alpha}\Pi(\mathbf{w}, \Theta)\Pi(\mathbf{w}, \Theta), & \mathcal{N}_\xi^\alpha(\mathbf{w}, p) = w^3 \frac{\partial w^\alpha}{\partial \xi}, \\ N_x^3(\mathbf{w}, \mathbf{w}) = w^\beta \frac{\partial w^3}{\partial x^\beta} + \varepsilon^{-1}w^\beta w^\lambda \Theta_{\beta\lambda} + (r\varepsilon)^{-1}\Pi(\mathbf{w}, \Theta)(2w^2 + r^2\Theta_2\Pi(\mathbf{w}, \Theta)), \\ \mathcal{N}_\xi^3(\mathbf{w}, p) = w^3 \frac{\partial w^3}{\partial \xi}, \end{cases} \quad (A.II.2)$$

and

$$\begin{cases} l^\alpha(\mathbf{w}, \Theta) &= -2r\varepsilon\delta_{2\alpha}\Theta_\lambda\frac{\partial w^3}{\partial x^\lambda} + \frac{1}{r}\frac{\partial w^\alpha}{\partial r} - \frac{w^2}{r^2}\delta_{2\alpha} + \delta_{2\alpha}(r\tilde{\Delta}\Theta - 2a\Theta_2)\Pi(\mathbf{w}, \Theta) \\ &\quad + B_\sigma^\alpha(\Theta)w^\sigma, \\ l_\xi^\alpha(\mathbf{w}, \Theta) &= (r\varepsilon)^{-2}a\frac{\partial w^\alpha}{\partial \xi} - 2\varepsilon^{-1}\Theta_\beta\frac{\partial w^\alpha}{\partial x^\beta} \\ &\quad - [(r\varepsilon)^{-1}(\delta_{\alpha\lambda}\Theta_2 + 2\delta_{2\alpha}\Theta_\lambda) + \varepsilon^{-1}\delta_{\alpha\lambda}\tilde{\Delta}\Theta]w^\lambda - 2r^{-1}\delta_{2\alpha}w^3, \\ B_\sigma^\alpha(\Theta) &= \delta_{2\alpha}[2(\delta_{2\sigma}|\tilde{\nabla}\Theta|^2 - r\Theta_\lambda\Theta_{\lambda\sigma})], \end{cases} \quad (A.II.3)$$

$$\begin{cases} l^3(\mathbf{w}, \Theta) &= (r\varepsilon)^{-1}(\delta_{2\beta}\Theta_\lambda + r\Theta_{\beta\lambda})\frac{\partial w^\lambda}{\partial x^\beta} + \frac{2}{r}\frac{\partial w^3}{\partial r} + \frac{\partial}{\partial x^\beta}(\varepsilon^{-1}\Theta_{\beta\sigma}w^\sigma) \\ &\quad + a\Theta_2\Theta_2w^3 + B_\sigma^3(\Theta)w^\sigma, \\ l_\xi^3(\mathbf{w}, \Theta) &= (r\varepsilon)^{-2}a\frac{\partial w^3}{\partial \xi} - 2\varepsilon^{-1}\Theta_\beta\frac{\partial w^3}{\partial x^\beta} + 2\varepsilon^{-2}(r^{-3}\delta_{2\sigma} - \Theta_\beta\Theta_{\beta\sigma})w^\sigma \\ &\quad + \varepsilon^{-1}(r\Theta_2|\tilde{\nabla}\Theta|^2 - \tilde{\Delta}\Theta)w^3, \\ B_\sigma^3(\Theta) &= (r\varepsilon)^{-1}[(r^{-1} + ra\Theta_2\Theta_2)\Theta_\sigma + 2\Theta_{2\sigma}], \end{cases} \quad (A.II.4)$$

$$\begin{cases} \Theta_\alpha = \frac{\partial \Theta}{\partial x^\alpha}, & \Theta_{\alpha\beta} = \frac{\partial^2 \Theta}{\partial x^\alpha \partial x^\beta}, & \Pi(\mathbf{w}, \Theta) = \varepsilon w^3 + w^\lambda \Theta_\lambda, \\ \tilde{\Delta}\Theta = \Theta_{\alpha\alpha} = \Theta_{11} + \Theta_{22}, & |\tilde{\nabla}\Theta|^2 = \Theta_1^2 + \Theta_2^2, \end{cases} \quad (A.II.5)$$

Proof: Firstly, we derive the Trace Laplacian operator in the New Coordinate System, that is,

$$\begin{cases} \Delta w^\alpha = g^{ij}\nabla_i\nabla_j w^\alpha = \tilde{\Delta}w^\alpha + \frac{\partial}{\partial \xi}l_\xi^\alpha(\mathbf{w}, \Theta) + l^\alpha(\mathbf{w}, \Theta), \\ \Delta w^3 = g^{ij}\nabla_i\nabla_j w^3 = \tilde{\Delta}w^3 + \partial_\xi l_\xi^3(\mathbf{w}, \Theta) + l^3(\mathbf{w}, \Theta). \end{cases} \quad (A.II.6)$$

In fact, we have

$$\Delta w^\alpha = g^{ij}\nabla_i\nabla_j w^\alpha = \nabla_\beta\nabla_\beta w^\alpha - \varepsilon^{-1}\Theta_\beta(\nabla_3\nabla_\beta w^\alpha + \nabla_\beta\nabla_3 w^\alpha) + (r\varepsilon)^{-2}a\nabla_3\nabla_3 w^\alpha.$$

Employing (A.I.23) and (A.I.24) we claim that

$$\begin{aligned} \nabla_\beta\nabla_\beta w^\alpha &= \partial_\beta(\nabla_\beta w^\alpha) - \Gamma_{\beta\beta}^m\nabla_m w^\alpha + \Gamma_{\beta m}^\alpha\nabla_\beta w^m \\ &= \partial_\beta(\frac{\partial w^\alpha}{\partial x^\beta} - r\delta_{2\alpha}\Theta_\beta\Pi(\mathbf{w}, \Theta)) + (\Gamma_{\beta\sigma}^\alpha\delta_\beta^\lambda - \Gamma_{\beta\beta}^\lambda\delta_{\alpha\sigma})\nabla_\lambda w^\sigma \\ &\quad + \Gamma_{3\beta}^\alpha\nabla_\beta w^3 - \Gamma_{\beta\beta}^3\nabla_3 w^\alpha, \\ \nabla_\beta\nabla_3 w^\alpha &= \partial_\beta\nabla_3 w^\alpha - \Gamma_{3\beta}^\lambda\nabla_\lambda w^\alpha + (\Gamma_{\lambda\beta}^\alpha - \Gamma_{3\beta}^3\delta_{\alpha\lambda})\nabla_3 w^\lambda + \Gamma_{3\beta}^\alpha\nabla_3 w^3, \\ \nabla_3\nabla_\beta w^\alpha &= \partial_\xi\nabla_\beta w^\alpha + (\Gamma_{3\sigma}^\alpha\delta_{\beta\lambda} - \Gamma_{3\beta}^\lambda\delta_{\alpha\sigma})\nabla_\lambda w^\sigma - \Gamma_{3\beta}^3\nabla_3 w^\alpha + \Gamma_{33}^\alpha\nabla_\beta w^3, \\ \nabla_3\nabla_3 w^\alpha &= \partial_\xi\nabla_3 w^\alpha + (\Gamma_{3\lambda}^\alpha - \Gamma_{33}^3\delta_{\alpha\lambda})\nabla_3 w^\lambda + \Gamma_{33}^\alpha\nabla_3 w^3 - \Gamma_{33}^\beta\nabla_\beta w^\alpha, \\ &\quad - \varepsilon^{-1}\Theta_\beta(\nabla_3\nabla_\beta w^\alpha + \nabla_\beta\nabla_3 w^\alpha) + (r\varepsilon)^{-2}a\nabla_3\nabla_3 w^\alpha \\ &= (r\varepsilon)^{-2}a\partial_\xi\nabla_3 w^\alpha - \varepsilon^{-1}\Theta_\beta(\partial_\beta\nabla_3 w^\alpha + \partial_\xi\nabla_\beta w^\alpha) \\ &\quad + [-(r\varepsilon)^{-2}a\Gamma_{33}^\lambda\delta_{\alpha\sigma} + \varepsilon^{-1}\Theta_\beta(2\Gamma_{3\beta}^\lambda\delta_{\alpha\sigma} - \Gamma_{3\sigma}^\alpha\delta_{\beta\lambda})]\nabla_\lambda w^\sigma \\ &\quad + [(r\varepsilon)^{-2}a(\Gamma_{3\lambda}^\alpha - \Gamma_{33}^3\delta_{\alpha\lambda}) + \varepsilon^{-1}\Theta_\beta(2\Gamma_{3\beta}^3\delta_{\alpha\lambda} - \Gamma_{\lambda\beta}^\alpha)]\nabla_3 w^\lambda \\ &\quad - \varepsilon^{-1}\Theta_\beta\Gamma_{33}^\alpha\nabla_\beta w^3 + [(r\varepsilon)^{-2}a\Gamma_{33}^\alpha - \varepsilon^{-1}\Theta_\beta\Gamma_{3\beta}^\alpha]\nabla_3 w^3 \end{aligned}$$

and

$$\begin{aligned} (r\varepsilon)^{-2}a\partial_\xi\nabla_3 w^\alpha - \varepsilon^{-1}\Theta_\beta(\partial_\beta\nabla_3 w^\alpha + \partial_\xi\nabla_\beta w^\alpha) \\ = \frac{\partial}{\partial \xi}[(r\varepsilon)^{-2}a\frac{\partial w^\alpha}{\partial \xi} - (r\varepsilon)^{-1}\delta_{2\alpha}\Pi(\mathbf{w}, \Theta) - 2\varepsilon^{-1}\Theta_\beta\frac{\partial w^\alpha}{\partial x^\beta}] + \Theta_\beta\delta_{2\alpha}\frac{\partial}{\partial x^\beta}(r\Pi(\mathbf{w}, \Theta)), \end{aligned}$$

Summing up the above results, we get

$$\begin{aligned} \Delta w^\alpha &= \partial_\beta\partial_\beta w^\alpha + \frac{\partial}{\partial \xi}[(r\varepsilon)^{-2}a\frac{\partial w^\alpha}{\partial \xi} - (r\varepsilon)^{-1}\delta_{2\alpha}\Pi(\mathbf{w}, \Theta) - 2\varepsilon^{-1}\Theta_\beta\frac{\partial w^\alpha}{\partial x^\beta}] \\ &\quad - \delta_{2\alpha}r\tilde{\Delta}\Theta\Pi(\mathbf{w}, \Theta) + I_1\nabla_\lambda w^\sigma + I_2\nabla_3 w^\lambda + I_3\nabla_\lambda w^3 + I_4\nabla_3 w^3 \\ &= \tilde{\Delta}w^\alpha + \frac{\partial}{\partial \xi}[(r\varepsilon)^{-2}a\frac{\partial w^\alpha}{\partial \xi} - (r\varepsilon)^{-1}\delta_{2\alpha}\Pi(\mathbf{w}, \Theta) - 2\varepsilon^{-1}\Theta_\beta\frac{\partial w^\alpha}{\partial x^\beta} + I_2w^\lambda + I_4w^3] \\ &= +I_1\frac{\partial w^\sigma}{\partial x^\lambda} + I_3\frac{\partial w^3}{\partial x^\lambda} + [-r\delta_{2\sigma}\Theta_\lambda I_1 - r\varepsilon\delta_{2\lambda}I_2 + (r\varepsilon)^{-1}a_{2\lambda}I_3 + r\Theta_2 I_4]\Pi(\mathbf{w}, \Theta) \\ &= +[\varepsilon^{-1}(\Theta_{\lambda\sigma} - r^{-1}\Theta_\lambda\delta_{2\sigma})I_3 + r^{-1}\delta_{2\sigma}I_4]w^\sigma, \end{aligned} \quad (A.II.7)$$

where some marked symbols are

$$\begin{aligned}
I_1 &= [\Gamma_{\lambda\sigma}^\alpha - \Gamma_{\beta\beta}^\lambda \delta_{\alpha\sigma} + \varepsilon^{-1} \Theta_\beta (2\Gamma_{3\beta}^\lambda \delta_{\alpha\sigma} - \Gamma_{3\sigma}^\alpha \delta_{\beta\lambda}) - (r\varepsilon)^{-2} a \Gamma_{33}^\lambda \delta_{\alpha\sigma}] \\
&= -r\delta_{2\alpha} \Theta_\lambda \Theta_\sigma + r\delta_{2\lambda} |\tilde{\nabla} \Theta|^2 \delta_{\alpha\sigma} + \varepsilon^{-1} \Theta_\beta (-r\varepsilon) (2\delta_{2\lambda} \Theta_\beta \delta_{\alpha\sigma} - \delta_{2\alpha} \Theta_\sigma \delta_{\beta\lambda}) \\
&\quad - (r\varepsilon)^{-2} a (-r\varepsilon^2) \delta_{2\lambda} \delta_{\alpha\sigma} = r^{-1} \delta_{2\lambda} \delta_{\alpha\sigma}, \\
I_2 &= -\Gamma_{\beta\beta}^3 \delta_{\alpha\lambda} + \varepsilon^{-1} \Theta_\beta (2\Gamma_{3\beta}^3 \delta_{\alpha\lambda} - \Gamma_{\beta\lambda}^\alpha) + (r\varepsilon)^{-2} a (\Gamma_{3\lambda}^\alpha - \Gamma_{33}^3 \delta_{\alpha\lambda}) \\
&= -[(r\varepsilon)^{-1} (a_{2\beta} \Theta_\beta + \Theta_2) + \varepsilon^{-1} \tilde{\Delta} \Theta] \delta_{\alpha\lambda} \\
&\quad + \varepsilon^{-1} \Theta_\beta (2r^{-1} a_{2\beta} \delta_{\alpha\lambda} + r\delta_{2\alpha} \Theta_\lambda \Theta_\beta) + (r\varepsilon)^{-2} a (-r\varepsilon \delta_{2\alpha} \Theta_\lambda - r\varepsilon \Theta_2 \delta_{\alpha\lambda}) \\
&= -(r\varepsilon)^{-1} (\delta_{\alpha\lambda} \Theta_2 + \delta_{2\alpha} \Theta_\lambda) - \varepsilon^{-1} \delta_{\alpha\lambda} \tilde{\Delta} \Theta, \\
I_3 &= \Gamma_{3\lambda}^\alpha - \varepsilon^{-1} \Theta_\lambda \Gamma_{33}^\alpha = -r\varepsilon \delta_{2\alpha} \Theta_\lambda - \varepsilon^{-1} \Theta_\lambda (-\varepsilon^2 r \delta_{2\alpha}) = -2r\varepsilon \delta_{2\alpha} \Theta_\lambda, \\
I_4 &= (r\varepsilon)^{-2} a \Gamma_{33}^\alpha - \varepsilon^{-1} \Theta_\beta \Gamma_{3\beta}^\alpha = (r\varepsilon)^{-2} a (-r\varepsilon^2 \delta_{2\alpha}) - \varepsilon^{-1} \Theta_\beta (-r\varepsilon \delta_{2\alpha} \Theta_\beta) = -r^{-1} \delta_{2\alpha},
\end{aligned}$$

Therefore

$$\begin{aligned}
&-r\delta_{2\sigma} \Theta_\lambda I_1 - r\varepsilon \delta_{2\lambda} I_2 + (r\varepsilon)^{-1} a_{2\lambda} I_3 + r\Theta_2 I_4 = (r\tilde{\Delta} \Theta - 2a\Theta_2) \delta_{2\alpha}, \\
&\varepsilon^{-1} (\Theta_{\lambda\sigma} - r^{-1} \Theta_\lambda \delta_{2\sigma}) I_3 + r^{-1} \delta_{2\sigma} I_4 = [r^{-2} (2a - 3) \delta_{2\sigma} - 2r\Theta_\lambda \Theta_{\lambda\sigma}] \delta_{2\alpha}.
\end{aligned}$$

Substituting the above expression into (A.II.7), then

$$\begin{aligned}
\Delta w^\alpha &= \tilde{\Delta} w^\alpha + \frac{\partial}{\partial \xi} [(r\varepsilon)^{-2} a \frac{\partial w^\alpha}{\partial \xi} - (r\varepsilon)^{-1} \delta_{2\alpha} \Pi(w, \Theta) - 2\varepsilon^{-1} \Theta_\beta \frac{\partial w^\alpha}{\partial x^\beta} + I_2 w^\lambda + I_4 w^3] \\
&\quad + r^{-1} \frac{\partial w^\alpha}{\partial r} - 2r\varepsilon \delta_{2\alpha} \Theta_\lambda \frac{\partial w^3}{\partial x^\lambda} + [(r\tilde{\Delta} \Theta - 2a\Theta_2) \delta_{2\alpha}] \Pi(w, \Theta) \\
&\quad + [r^{-2} (2a - 3) \delta_{2\sigma} - 2r\Theta_\lambda \Theta_{\lambda\sigma}] \delta_{2\alpha} w^\sigma.
\end{aligned} \tag{A.II.8}$$

(A.II.8) can be rewritten in a splitting form, that is

$$\Delta w^\alpha = \tilde{\Delta} w^\alpha + \frac{\partial}{\partial \xi} l_\xi^\alpha(\mathbf{w}, \Theta) + l^\alpha(\mathbf{w}, \Theta), \tag{A.II.9}$$

where $l^\alpha(w, \Theta)$, $l_\xi^\alpha(w, \Theta)$, $B_\sigma^\alpha(\Theta)$ are formulated in (A.II.3).

Our task is now to prove the second equality of (A.II.6). Indeed,

$$\Delta w^3 = \nabla_\beta \nabla_\beta w^3 - \varepsilon^{-1} \Theta_\beta (\nabla_\beta \nabla_3 w^3 + \nabla_3 \nabla_\beta w^3) + (r\varepsilon)^{-2} a \nabla_3 \nabla_3 w^3.$$

An argument similar to the one used in proof of the first equality of (A.II.6) shows that

$$\begin{aligned}
\Delta w^3 &= \partial_\beta (\nabla_\beta w^3) + \Gamma_{\beta\lambda}^3 \nabla_\beta w^\lambda + (\Gamma_{3\beta}^3 - \Gamma_{\lambda\lambda}^\beta) \nabla_\beta w^3 - \Gamma_{\beta\beta}^3 \nabla_3 w^3 \\
&\quad - \varepsilon^{-1} \Theta_\beta [\partial_\beta \nabla_3 w^3 + \Gamma_{\beta\lambda}^3 \nabla_3 w^\lambda + \Gamma_{3\beta}^3 \nabla_3 w^3 - \Gamma_{3\beta}^\lambda \nabla_\lambda w^3 - \Gamma_{3\beta}^3 \nabla_3 w^3] \\
&\quad + \partial_\xi \nabla_\beta w^3 + \Gamma_{3\lambda}^3 \nabla_\beta w^\lambda + (\Gamma_{33}^3 \delta_{\beta\lambda} - \Gamma_{3\beta}^\lambda) \nabla_\lambda w^3 - \Gamma_{3\beta}^3 \nabla_3 w^3 \\
&\quad + (r\varepsilon)^{-2} a [\partial_\xi \nabla_3 w^3 + \Gamma_{3\lambda}^3 \nabla_3 w^\lambda + \Gamma_{33}^3 \nabla_3 w^3 - \Gamma_{33}^\lambda \nabla_\lambda w^3 - \Gamma_{33}^3 \nabla_3 w^3],
\end{aligned}$$

and

$$\begin{aligned}
\Delta w^3 &= \partial_\beta (\nabla_\beta w^3) - \varepsilon^{-1} \Theta_\beta \partial_\beta \nabla_3 w^3 + \frac{\partial}{\partial \xi} [(r\varepsilon)^{-2} a \nabla_3 w^3 - \varepsilon^{-1} \Theta_\beta \nabla_\beta w^3] \\
&\quad + J_1 \nabla_\beta w^\lambda + J_2 \nabla_\beta w^3 + J_3 \nabla_3 w^\lambda + J_4 \nabla_3 w^3,
\end{aligned} \tag{A.II.10}$$

where (by (A.I.23))

$$\begin{cases} J_1 = (\Gamma_{\beta\lambda}^3 - \varepsilon^{-1} \Theta_\beta \Gamma_{3\lambda}^3) = (r\varepsilon)^{-1} (\delta_{2\beta} \Theta_\lambda + r\Theta_{\beta\lambda}), \\ J_2 = \Gamma_{3\beta}^3 - \Gamma_{\lambda\lambda}^\beta - (r\varepsilon)^{-2} a \Gamma_{33}^\beta + \varepsilon^{-1} \Theta_\lambda (-\Gamma_{33}^3 \delta_{\beta\lambda} + 2\Gamma_{3\lambda}^\beta) = 2r^{-1} \delta_{2\beta}, \\ J_3 = (r\varepsilon)^{-2} a \Gamma_{3\lambda}^3 - \varepsilon^{-1} \Theta_\beta \Gamma_{\beta\lambda}^3 = \varepsilon^{-2} r^{-3} (\delta_{2\lambda} - r^3 \Theta_\beta \Theta_{\beta\lambda}), \\ J_4 = -\Gamma_{\beta\beta}^3 + 2\varepsilon^{-1} \Theta_\beta \Gamma_{3\beta}^3 = (\varepsilon)^{-1} (r\Theta_2 |\tilde{\nabla} \Theta|^2 - \tilde{\Delta} \Theta) \end{cases} \tag{A.II.11}$$

Some further calculations show that

$$\begin{aligned}
& \partial_\beta(\nabla_\beta w^3) - \varepsilon^{-1}\Theta_\beta\partial_\beta\nabla_3w^3 + \frac{\partial}{\partial\xi}[(r\varepsilon)^{-2}a\nabla_3w^3 - \varepsilon^{-1}\Theta_\beta\nabla_\beta w^3] \\
&= \partial_\beta(\frac{\partial w^3}{\partial x^\beta} + (r\varepsilon)^{-1}[(\delta_{\alpha 2}\Theta_\beta + r\Theta_{\alpha\beta})w^\alpha + a_{2\beta}\Pi(w, \Theta)]) \\
&\quad - \varepsilon^{-1}\Theta_\beta\partial_\beta(\frac{\partial w^3}{\partial\xi} + \frac{w^2}{r} + r\Theta_2\Pi(w, \Theta)) + \frac{\partial}{\partial\xi}[(r\varepsilon)^{-2}a\nabla_3w^3 - \varepsilon^{-1}\Theta_\beta\nabla_\beta w^3] \\
&= \tilde{\Delta}w^3 + \frac{\partial}{\partial\xi}[(r\varepsilon)^{-2}a\nabla_3w^3 - \varepsilon^{-1}\Theta_\beta\nabla_\beta w^3 - \varepsilon^{-1}\Theta_\beta\frac{\partial w^3}{\partial x^\beta}] \\
&\quad + \frac{\partial}{\partial x^\beta}((r\varepsilon)^{-1}[(\delta_{\alpha 2}\Theta_\beta + r\Theta_{\alpha\beta})w^\alpha + a_{2\beta}\Pi(w, \Theta)] - \varepsilon^{-1}\Theta_\beta(\frac{w^2}{r} + r\Theta_2\Pi(w, \Theta))) \\
&\quad + \varepsilon^{-1}\tilde{\Delta}\Theta(\frac{w^2}{r} + r\Theta_2\Pi(w, \Theta)).
\end{aligned}$$

Combining like terms, we get

$$\begin{aligned}
& \partial_\beta(\nabla_\beta w^3) - \varepsilon^{-1}\Theta_\beta\partial_\beta\nabla_3w^3 + \frac{\partial}{\partial\xi}[(r\varepsilon)^{-2}a\nabla_3w^3 - \varepsilon^{-1}\Theta_\beta\nabla_\beta w^3] \\
&= \tilde{\Delta}w^3 + \frac{\partial}{\partial\xi}[(r\varepsilon)^{-2}a\nabla_3w^3 - \varepsilon^{-1}\Theta_\beta\nabla_\beta w^3 - \varepsilon^{-1}\Theta_\beta\frac{\partial w^3}{\partial x^\beta}] \\
&\quad + \frac{\partial}{\partial x^\beta}(\varepsilon^{-1}\Theta_{\alpha\beta}w^\alpha) + \varepsilon^{-1}\tilde{\Delta}\Theta(\frac{w^2}{r} + r\Theta_2\Pi(w, \Theta)).
\end{aligned}$$

On the other hand, by using (A.I.24), then

$$\begin{aligned}
& J_1\nabla_\beta w^\lambda + J_2\nabla_\beta w^3 + J_3\nabla_3w^\lambda + J_4\nabla_3w^3 = J_1\partial_\beta w^\lambda + J_2\partial_\beta w^3 + \partial_\xi(J_3w^\lambda + J_4w^3) \\
&\quad + [-r\delta_{2\lambda}\Theta_\beta J_1 + (r\varepsilon)^{-1}a_{2\beta}J_2 - r\varepsilon\delta_{2\lambda}J_3 + r\Theta_2J_4]\Pi(w, \Theta) + (r\varepsilon)^{-1}(2\Theta_{2\alpha} - \tilde{\Delta}\Theta\delta_{2\alpha})w^\alpha \\
&= J_1\partial_\beta w^\lambda + J_2\partial_\beta w^3 + \partial_\xi(J_3w^\lambda + J_4w^3) \\
&\quad + [\varepsilon^{-1}r^{-2} + \varepsilon^{-1}a\Theta_2\Theta_2 - r\varepsilon^{-1}\Theta_2\tilde{\Delta}\Theta]\Pi(w, \Theta) + (r\varepsilon)^{-1}(2\Theta_{2\alpha} - \tilde{\Delta}\Theta\delta_{2\alpha})w^\alpha.
\end{aligned}$$

Summing up the above conclusions, (A.II.11) becomes

$$\begin{aligned}
\Delta w^3 = & \tilde{\Delta}w^3 + \frac{\partial}{\partial\xi}[(r\varepsilon)^{-2}a\nabla_3w^3 - \varepsilon^{-1}\Theta_\beta\nabla_\beta w^3 - \varepsilon^{-1}\Theta_\beta\frac{\partial w^3}{\partial x^\beta} + J_3w^\lambda + J_4w^3] \\
& + \frac{\partial}{\partial x^\beta}(\varepsilon^{-1}\Theta_{\alpha\beta}w^\alpha) + \varepsilon^{-1}\tilde{\Delta}\Theta(\frac{w^2}{r} + r\Theta_2\Pi(w, \Theta)) + J_1\partial_\beta w^\lambda + J_2\partial_\beta w^3 \\
& + [\varepsilon^{-1}r^{-2} + \varepsilon^{-1}a\Theta_2\Theta_2 - r\varepsilon^{-1}\Theta_2\tilde{\Delta}\Theta]\Pi(w, \Theta) + (r\varepsilon)^{-1}(2\Theta_{2\alpha} - \tilde{\Delta}\Theta\delta_{2\alpha})w^\alpha.
\end{aligned} \tag{A.II.12}$$

Thanks to the expanded formula

$$J_1\partial_\beta w^\lambda + J_2\partial_\beta w^3 = (r\varepsilon)^{-1}(\delta_{2\beta}\Theta_\lambda + r\Theta_{\beta\lambda})\frac{\partial w^\lambda}{\partial x^\beta} + \frac{2}{r}\frac{\partial w^3}{\partial r}.$$

Hence, similarly,

$$\Delta w^3 = \tilde{\Delta}w^3 + \frac{\partial}{\partial\xi}l_\xi^3(w, \Theta) + l^3(w, \Theta), \tag{A.II.13}$$

where $l^3(w, \Theta), l_\xi^3(w, \Theta)$ are expressed in (A.II.4). This is the second expression of (A.II.6).

Our goal now is to consider the terms of the pressure. Actually, we have

$$g^{\alpha\beta}\partial_\beta p + g^{\alpha 3}\partial_\xi p = \delta^{\alpha\beta}\partial_\beta p - \varepsilon^{-1}\Theta_\alpha\partial_\xi p, \quad g^{3\alpha}\partial_\alpha p + g^{33}\partial_\xi p = -\varepsilon^{-1}\Theta_\alpha\partial_\alpha p + (r\varepsilon)^{-2}a\partial_\xi p. \tag{A.II.14}$$

By Using (A.I.24), the nonlinear terms are formulated as

$$\begin{cases} w^j\nabla_j w^\alpha &= w^\beta\nabla_\beta w^\alpha + w^3\nabla_3w^\alpha = w^\beta\frac{\partial w^\alpha}{\partial x^\beta} + w^3\frac{\partial w^\alpha}{\partial\xi} - r\delta_{2\alpha}\Pi(w, \Theta)\Pi(w, \Theta) \\ w^j\nabla_j w^3 &= w^\beta\frac{\partial w^3}{\partial x^\beta} + w^3\frac{\partial w^3}{\partial\xi} + \varepsilon^{-1}w^\beta w^\lambda\Theta_{\beta\lambda} \\ &\quad + (r\varepsilon)^{-1}\Pi(w, \Theta)[2w^2 + (x^2)^2\Theta_2\Pi(w, \Theta)], \end{cases} \tag{A.II.15}$$

Combing (A.II.6) with (A.II.14)-(A.II.15) obtains (A.II.1). The proof is completed. \square

A.3 The equations for the Gâteaux derivative of the solutions of NSEs

In this section we consider the Gâteaux derivatives of the solution of NSE with respect to the two dimensional manifold \mathfrak{S} which is a portion of the solid boundary of the flow passage in impeller.

Proposition A.4. Assume that there exists a Gâteaux derivatives ($\widehat{\mathbf{w}} := \frac{\mathcal{D}\mathbf{w}}{\mathcal{D}\Theta}, \widehat{p} := \frac{\mathcal{D}p}{\mathcal{D}\Theta}$) of the solutions (\mathbf{w}, p) of the Navier-Stokes equations (A.II.13) with boundary conditions

$$\mathbf{w}|_{\Gamma_s} = \mathbf{0}, \quad [\nu \frac{\partial \mathbf{w}}{\partial \mathbf{n}} - p\mathbf{n}]|_{\Gamma_1} = \mathbf{h}, \quad (\text{A.III.1})$$

where Γ_s and Γ_1 are boundary $\partial\Omega = \Gamma_s \cup \Gamma_1$, $\Omega = D \times [-1, 1]$. Then $(\widehat{\mathbf{w}}, \widehat{p})$ satisfy the following linearized Navier-Stokes equations with the corresponding homogenous boundary conditions, that is,

$$\begin{cases} \frac{\partial \widehat{w}^\alpha}{\partial x^\alpha} + \frac{\partial \widehat{w}^3}{\partial \xi} + \frac{\widehat{w}^2}{r} = 0, \\ -\nu \widetilde{\Delta} \widehat{w}^\alpha - \nu l^\alpha(\widehat{\mathbf{w}}, \Theta) + C^\alpha(\widehat{\mathbf{w}}, \boldsymbol{\omega}) + \nabla_\alpha \widehat{p} - \frac{\partial}{\partial \xi}(\nu l_\xi^\alpha(\widehat{\mathbf{w}}, \Theta) + \varepsilon^{-1} \Theta_\alpha \widehat{p}) + N_x^\alpha(\mathbf{w}, \widehat{\mathbf{w}}) \\ \quad + N_x^\alpha(\widehat{\mathbf{w}}, \mathbf{w}) + N_\xi^\alpha(\mathbf{w}, \widehat{\mathbf{w}}) + N_\xi^\alpha(\widehat{\mathbf{w}}, \mathbf{w}) + R^\alpha(\mathbf{w}, p, \Theta) = 0, \\ -\nu \widetilde{\Delta} \widehat{w}^3 - \nu l^3(\widehat{\mathbf{w}}, \Theta) + C^3(\widehat{\mathbf{w}}, \boldsymbol{\omega}) - \varepsilon^{-1} \Theta_\beta \partial_\beta \widehat{p} + \frac{\partial}{\partial \xi}(-\nu l_\xi^3(\widehat{\mathbf{w}}, \Theta) + (r\varepsilon)^{-2} a \widehat{p} \\ \quad + N_x^3(\mathbf{w}, \widehat{\mathbf{w}}) + N_x^3(\widehat{\mathbf{w}}, \mathbf{w}) + N_\xi^3(\mathbf{w}, \widehat{\mathbf{w}}) + N_\xi^3(\widehat{\mathbf{w}}, \mathbf{w}) + R^3(\mathbf{w}, p, \Theta) = 0, \end{cases} \quad (\text{A.III.2})$$

and

$$\begin{cases} \widehat{\mathbf{w}} = \mathbf{0}, & \text{on } \Gamma_s \cap \xi = \xi_k, \\ \nu \frac{\partial \widehat{\mathbf{w}}}{\partial \mathbf{n}} - \widehat{p}\mathbf{n} = 0, & \text{on } \Gamma_{in} \cap \xi = \xi_k, \quad \nu \frac{\partial \widehat{\mathbf{w}}}{\partial \mathbf{n}} - \widehat{p}\mathbf{n} = 0, & \text{on } \Gamma_{out} \cap \xi = \xi_k, \end{cases} \quad (\text{A.III.3})$$

where

$$\begin{cases} R^\alpha(\mathbf{w}, p)\boldsymbol{\eta} = -\nu \frac{\mathcal{D}l^\alpha}{\mathcal{D}\Theta} \boldsymbol{\eta} - \frac{\partial}{\partial \xi} [\frac{\mathcal{D}l_\xi^\alpha}{\mathcal{D}\Theta} \boldsymbol{\eta} + \varepsilon^{-1} p \boldsymbol{\eta}_\alpha] + \frac{\partial C^\alpha}{\partial \Theta} \boldsymbol{\eta} - 2r\delta_{2\alpha} \Theta_\lambda \Pi(\mathbf{w}, \Theta) \boldsymbol{\eta}_\lambda, \\ R^3(\mathbf{w}, p)\boldsymbol{\eta} = -\nu \frac{\mathcal{D}l^3}{\mathcal{D}\Theta} \boldsymbol{\eta} + \frac{\partial}{\partial \xi} [-\frac{\mathcal{D}l_\xi^3}{\mathcal{D}\Theta} \boldsymbol{\eta} + (r\varepsilon)^{-2} 2r^2 \Theta_\lambda p \boldsymbol{\eta}_\lambda] + \frac{\partial C^3}{\partial \Theta} \boldsymbol{\eta} - \varepsilon^{-1} \boldsymbol{\eta}_\alpha \partial_\alpha p \\ \quad + \varepsilon^{-1} w^\lambda w^\sigma \boldsymbol{\eta}_{\lambda\sigma} + (r\varepsilon)^{-1} [w^\lambda (2w^2 + r^2 \Theta_2 \Pi(\mathbf{w}, \Theta)) \\ \quad + \Pi(\mathbf{w}, \Theta) (\delta_{2\lambda} r^2 \Pi(\mathbf{w}, \Theta) + r^2 \Theta_2 w^\lambda)] \boldsymbol{\eta}_\lambda, \end{cases} \quad (\text{A.III.4})$$

Proof: The Navier-Stokes equations (A.II.1) can be rewritten as

$$\begin{cases} \frac{\partial w^\alpha}{\partial x^\alpha} + \frac{w^2}{r} + \frac{\partial w^3}{\partial \xi} = 0, \\ \mathcal{N}^\alpha(\mathbf{w}, p, \Theta) \mathbf{e}_\alpha + \mathcal{N}^3(\mathbf{w}, p, \Theta) \mathbf{e}_3 = f^\alpha \mathbf{e}_\alpha + f^3 \mathbf{e}_3. \end{cases} \quad (\text{A.III.5})$$

Set Gâteaux derivative with respect with Θ along any director

$$\boldsymbol{\eta} \in \mathcal{W} := H^1(D) \cap \{H^1(D), \text{ with } \mathbf{w}|_{\Gamma_s} = \mathbf{0}, \quad [\nu \frac{\partial \mathbf{w}}{\partial \mathbf{n}} - p\mathbf{n}]|_{\Gamma_1} = \mathbf{0}\}$$

denoted by $\frac{\mathcal{D}}{\mathcal{D}\Theta} \boldsymbol{\eta}$. Then from (A.III.5) we assert

$$\begin{aligned} & \frac{\mathcal{D}}{\mathcal{D}\Theta} \mathcal{N}^\alpha(\mathbf{w}, p, \Theta) \mathbf{e}_\alpha \boldsymbol{\eta} + \frac{\mathcal{D}}{\mathcal{D}\Theta} \mathcal{N}^3(\mathbf{w}, p, \Theta) \mathbf{e}_3 \boldsymbol{\eta} + \mathcal{N}^\alpha(\mathbf{w}, p, \Theta) \frac{\mathcal{D}\mathbf{e}_\alpha}{\mathcal{D}\Theta} \boldsymbol{\eta} + \mathcal{N}^3(\mathbf{w}, p, \Theta) \frac{\mathcal{D}\mathbf{e}_3}{\mathcal{D}\Theta} \boldsymbol{\eta} \\ & = f^\alpha \frac{\mathcal{D}\mathbf{e}_\alpha}{\mathcal{D}\Theta} \mathbf{e}_\alpha \boldsymbol{\eta} + f^3 \frac{\mathcal{D}\mathbf{e}_3}{\mathcal{D}\Theta} \mathbf{e}_3 \boldsymbol{\eta}, \\ & \frac{\mathcal{D}}{\mathcal{D}\Theta} \mathcal{N}^\alpha(\mathbf{w}, p, \Theta) \mathbf{e}_\alpha + \frac{\mathcal{D}}{\mathcal{D}\Theta} \mathcal{N}^3(\mathbf{w}, p, \Theta) \mathbf{e}_3 + [\mathcal{N}^\alpha(\mathbf{w}, p, \Theta) - f^\alpha] \frac{\mathcal{D}\mathbf{e}_\alpha}{\mathcal{D}\Theta} + [\mathcal{N}^3(\mathbf{w}, p, \Theta) - f^3] \frac{\mathcal{D}\mathbf{e}_3}{\mathcal{D}\Theta} \mathbf{e}_3 = 0. \end{aligned}$$

Since Navier-Stokes equation (A.III.2), it yields

$$\begin{cases} \frac{\mathcal{D}}{\mathcal{D}\Theta} \mathcal{N}^\alpha(\mathbf{w}, p, \Theta) \boldsymbol{\eta} = \frac{\partial}{\partial \Theta} \mathcal{N}^\alpha(\mathbf{w}, p, \Theta) \boldsymbol{\eta} + \frac{\partial}{\partial \mathbf{w}} \mathcal{N}^\alpha(\mathbf{w}, p, \Theta) \widehat{\mathbf{w}} \boldsymbol{\eta} + \frac{\partial}{\partial p} \mathcal{N}^\alpha(\mathbf{w}, p, \Theta) \widehat{p} \boldsymbol{\eta} = 0, \\ \frac{\mathcal{D}}{\mathcal{D}\Theta} \mathcal{N}^3(\mathbf{w}, p, \Theta) \boldsymbol{\eta} = \frac{\partial}{\partial \Theta} \mathcal{N}^3(\mathbf{w}, p, \Theta) \boldsymbol{\eta} + \frac{\partial}{\partial \mathbf{w}} \mathcal{N}^3(\mathbf{w}, p, \Theta) \widehat{\mathbf{w}} \boldsymbol{\eta} + \frac{\partial}{\partial p} \mathcal{N}^3(\mathbf{w}, p, \Theta) \widehat{p} \boldsymbol{\eta} = 0, \end{cases} \quad (\text{A.III.8})$$

It is obvious that

$$R^\alpha(\mathbf{w}, \Theta) \boldsymbol{\eta} = \frac{\mathcal{D}}{\mathcal{D}\Theta} \mathcal{N}^\alpha(\mathbf{w}, p, \Theta) \boldsymbol{\eta}, \quad R^3(\mathbf{w}, \Theta) \boldsymbol{\eta} = \frac{\mathcal{D}}{\mathcal{D}\Theta} \mathcal{N}^3(\mathbf{w}, p, \Theta) \boldsymbol{\eta}.$$

□

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